9.3. Why the bit state space S_2 is a Euclidean ball

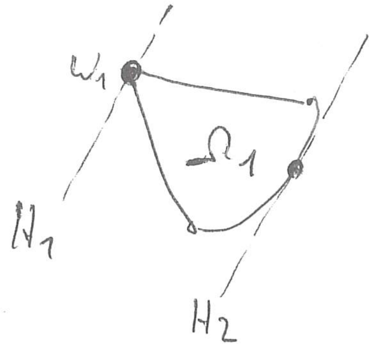
<u>Lemma</u>: The system S_1 contains only a single state, i.e. $\Omega_1 = \{\omega\}$.

True for "one-level" QT and CPT:

(1) is the unique 1×1 density matrix, and the unique single-entry probability vector.

<u>Proof</u>: Suppose that Ω_1 contains at least two different states, say, $\varphi_1 \neq \varphi_2 \in \Omega_1$. Then $\{\lambda \varphi_1 + (1-\lambda)\varphi_2 \mid 0 \leq \lambda \leq 1\} \subseteq \Omega_1$ and so $\dim \Omega_1 \geq 1$.

Convex geometry (Website) tells us that there exists an exposed point $\omega_1 \in \Omega_1$.



Let H_1 be the hyperplane (in the vector space V_1) such that $\omega_1 = H_1 \cap \Omega_1$.

$H_1 = \{x \in V_1 \mid (e^{(1)}, x) = 1\}$,

$H_2 = \{x \in V_1 \mid (e^{(1)}, x) = 0\}$

$(e^{(1)}, \omega) \in [0,1]$ for all $\omega \in \Omega_1$.

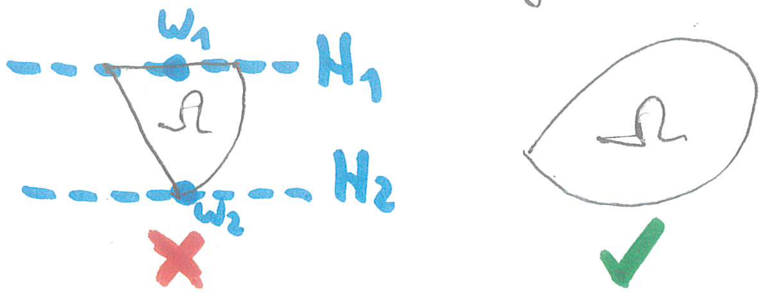
Such a functional exists because H_1 is a supporting hyperplane, and H_2 is a parallel supporting hyperplane.

But $e^{(1)}, u_1 - u^{(1)} =: e^{(2)}$ defines a valid measurement that distinguishes 2 states ω_1, ω_2 of Ω_1, where $\omega_2 \in \Omega_1 \cap H_2$.

This is a contradiction to $N_1 = 1$. □

uses no-restriction hypothesis.

Lemma: The bit state space Ω_2 is strictly convex, i.e. it does not contain any lines in its boundary.



"Strictly convex" is also equivalent to:

- all boundary points are extremal points (i.e. pure states),
- the only proper faces are the pure states.

Proof: [redacted] Suppose that Ω_2 is not strictly convex. Then we find a supporting hyperplane H_1 such that $H_1 \cap \Omega_2$ contains more than one point **(like in the figure above)**. Again, there is a measurement $e^{(1)}, e^{(2)}$ such that $H_i \cap \Omega_2 = \{\omega \in \Omega_2 \mid (e^{(i)}, \omega) = 1\}$.
According to the subspace axiom,

$$\{\omega \in \Omega_2 \mid (e^{(2)}, \omega) = 0\} = H_1 \cap \Omega_2$$

is equivalent to Ω_1. However, this set contains more than a single point, and $|\Omega_1| = 1$. This is a contradiction. $\square$

Lemma: Ω_2 is equivalent to a Euclidean ball of some dimension d.

Proof: Pick any pure state $\omega \in \Omega_2$, and define the "maximally mixed state"

$$\mu := \int_{\mathcal{G}_2} T\omega \, dT .$$

(recall: invariant measure on the group!)

Then $T\mu = T\int_{\mathcal{G}_2} T'\omega\, dT' = \int_{\mathcal{G}_2} TT'\omega\, dT' = \int_{\mathcal{G}_2} (TT')\omega\, d(TT')$

$= \mu$ for all $T\in\mathcal{G}_2$

(in QT: $\mu = \int_{SU(2)} U\varrho U^\dagger dU = \frac{1}{2}\cdot\mathbb{1}$, $U\mu U^\dagger = \mu$.)

For $\omega\in\Omega_2$, define the "Bloch vector" $\vec{\omega} := \omega - \mu \in \mathbb{R}^d$ where $d = \dim \Omega_2$.

$T\omega = \varphi \Rightarrow T\vec{\omega} = T(\omega-\mu) = T\omega - \mu = \varphi - \mu = \vec{\varphi}$

$\Rightarrow$ T acts on the "Bloch space" $\mathbb{R}^d$.

$$\langle \vec{x}, \vec{y}\rangle := \alpha \int_{\mathcal{G}_2} T\vec{x}\cdot T\vec{y}\, dT \quad (\alpha>0\text{: choose later}).$$

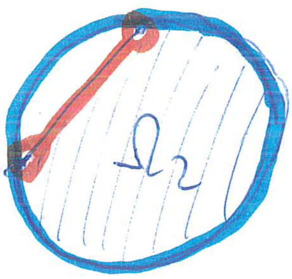
$\Rightarrow \langle T\vec{x}, T\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ for all $\vec{x},\vec{y}\in\mathbb{R}^d$, $T\in\mathcal{G}_2$

$\Rightarrow$ the T are orthogonal matrices.

If ω and φ are pure states, then the Reversibility principle says that there exists $T\in\mathcal{G}_2$ with $T\omega = \varphi$

$\Rightarrow \|\vec{\varphi}\|^2 = \langle\vec{\varphi},\vec{\varphi}\rangle = \langle T\vec{\omega}, T\vec{\omega}\rangle = \langle\vec{\omega},\vec{\omega}\rangle = \|\vec{\omega}\|^2.$

Fix $\alpha>0$ such that this equals 1.



Thus, Ω_2 is a subset of the Euclidean unit ball B^d in $(\mathbb{R}^d, \langle\cdot,\cdot\rangle)$, and all pure states are on the sphere.

Could $\Omega_2 \subsetneq B^d$? If so, then there would be **mixed states in the top. boundary of Ω_2**, which contradicts strict convexity. $\square$