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Bit ball: $d = K_2 - 1 \in \{1, 3, 7, 15, 31, \ldots\}$.

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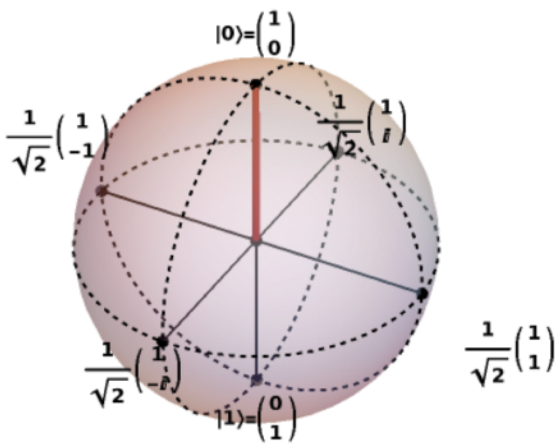

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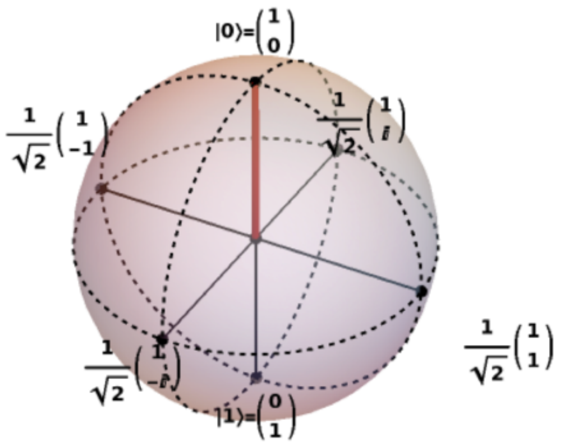

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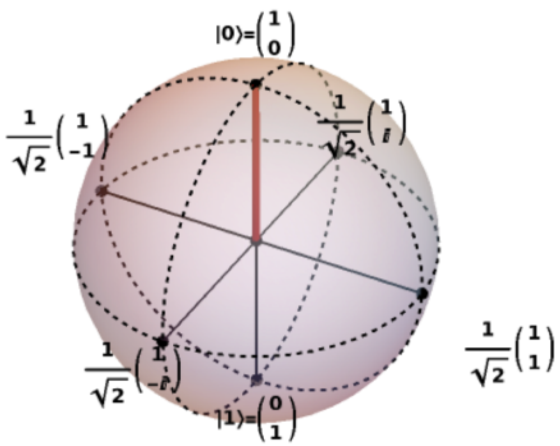

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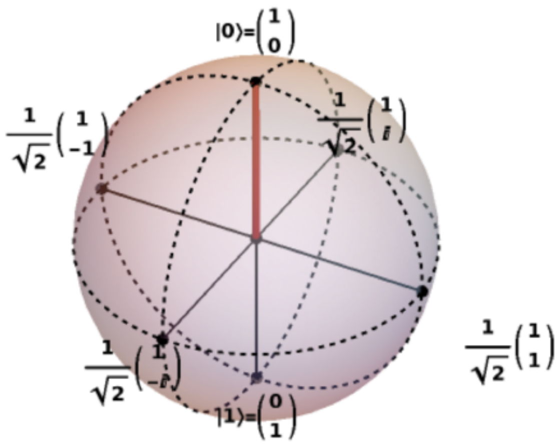

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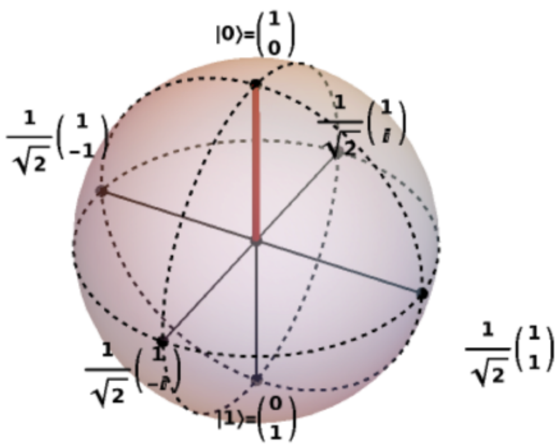

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This is an exercise in topology which we consider, for here, as settled.

Since $d \in \{1, 3, 7, 15, 31, \ldots\},$ the following turns out to be true for d>1:

- If $d \neq 7$ then we must have $\mathcal{T}_2 = \mathrm{SO}(d)$.
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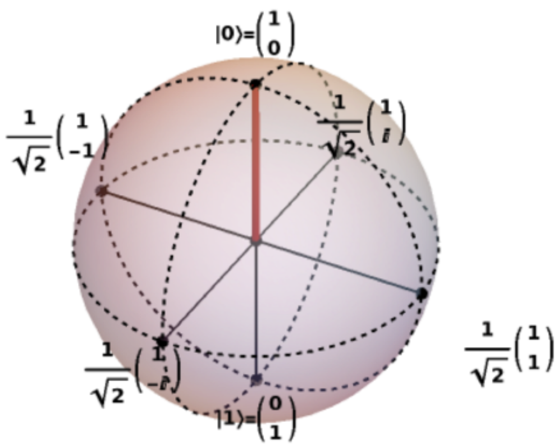

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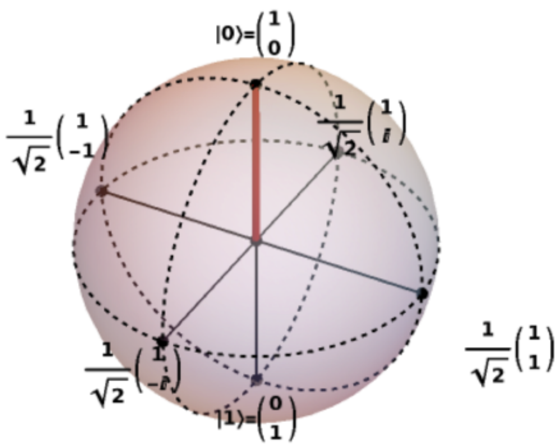

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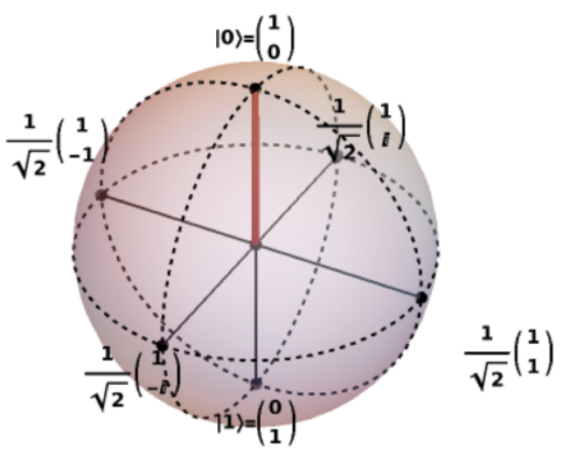

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4. On *two* such bits with perfectly distinguishable states $\omega_{ij} = \omega_i \otimes \omega_j$ $(i, j = 0, 1)$, the interplay of local independent transformations and global transformations on the "sub-bit" $\{\omega_{00}, \omega_{11}\}$ leads to contradiction if *d>3*.

“Why” is QT’s Bloch ball three-dimensional?

Group representation theory: $\mathrm{SO}(d-1)\otimes\mathrm{SO}(d-1)$ acts irreducibly on $\mathbb{R}^{d-1}\otimes\mathbb{R}^{d-1}$ if $d\geq 4.$

This follows from the fact (via character theory) that

$\mathrm{SO}(d-1)$ acts irreducibly on $\mathbb{C}^{d-1}$ if $d\geq 4.$

However, for *d=3*, the group $\mathrm{SO}(d-1)=\mathrm{SO}(2)$ is **Abelian**, and so all its irreducible representations are one-dimensional. Hence, it cannot act irreducibly on $\mathbb{C}^{d-1}=\mathbb{C}^2$. In fact,
$\left(\begin{array}{c}1\\ \pm i\end{array}\right)$ are eigenvectors of $\left(\begin{array}{cc}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{array}\right).$

Thus, the above contradiction does not occur. Hence, in some sense:

The Bloch ball is three-dimensional “because” $\mathrm{SO}(d-1)$ is non-trivial and Abelian only for *d=3*.