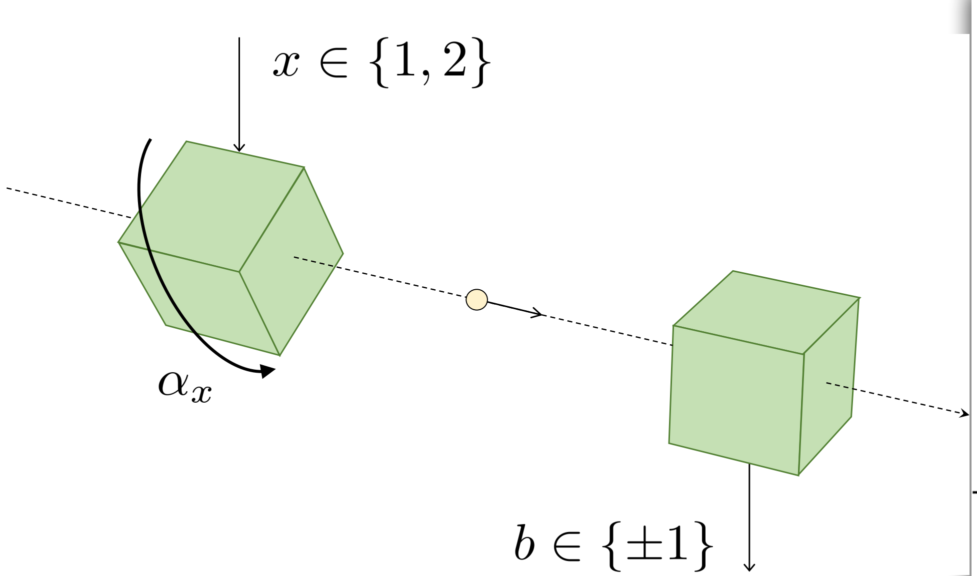
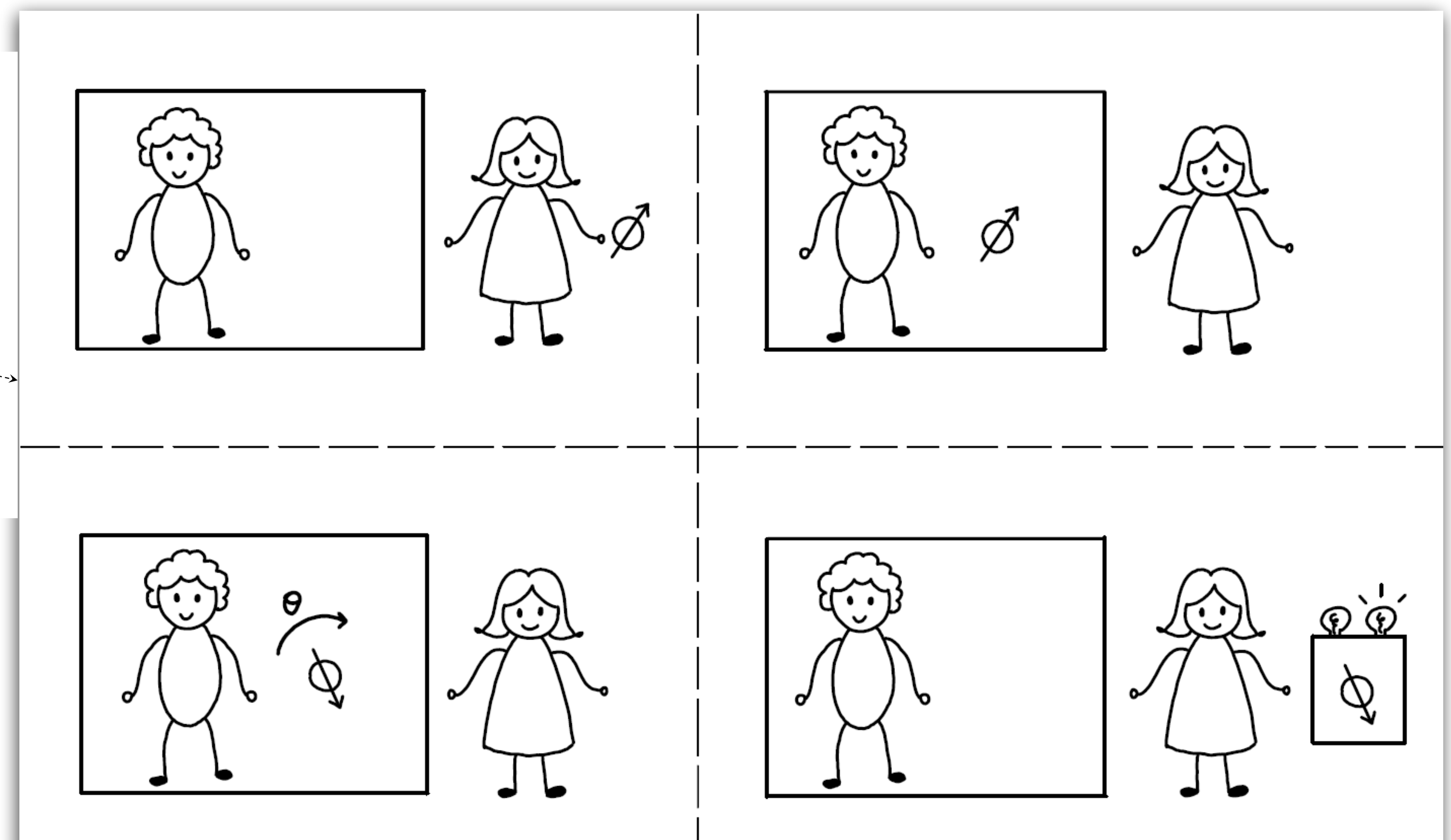
IQOQI – INSTITUTE FOR QUANTUM OPTICS AND QUANTUM INFORMATION VIENNA

Space, time and quantum probabilities:
from fundamental insights to protocols

Markus P. Müller

IQOQI Vienna & Perimeter Institute





Austrian Science Fund

Overview

1. Motivation (and some history)

2. “Rotation boxes” within and beyond QT

3. A metrological game and the (sub)optimality of QT

4. A theory-independent SDI randomness generator

5. Conclusions

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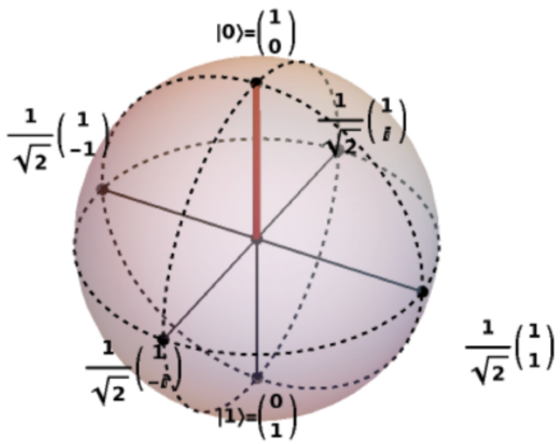
Invitation: the qubit and spatial rotations



Qubit Bloch ball

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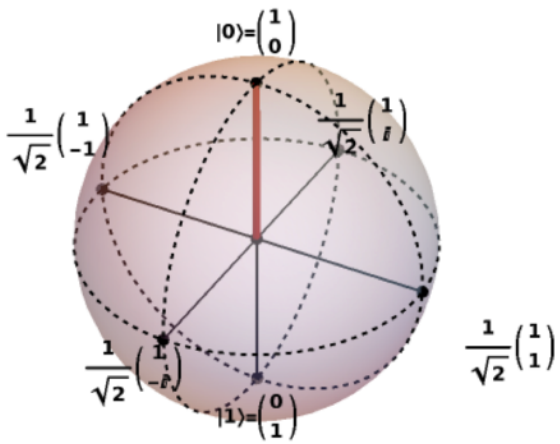

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$$\rho = \frac{1}{2}\mathbf{1} + \vec{r}\cdot\vec{\sigma} = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\mathrm{tr}(\rho) = 1, \quad \rho \geq 0 \Leftrightarrow |\vec{r}| \leq 1.$$

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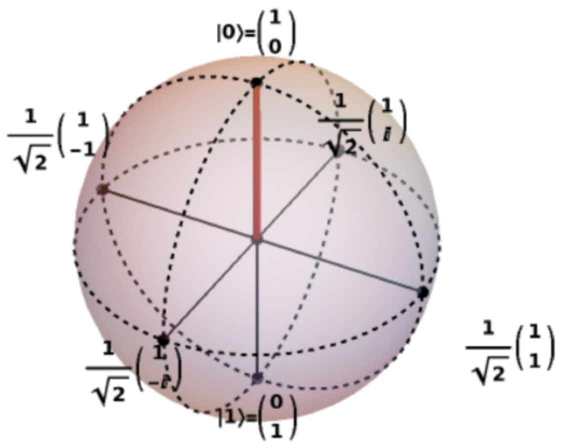

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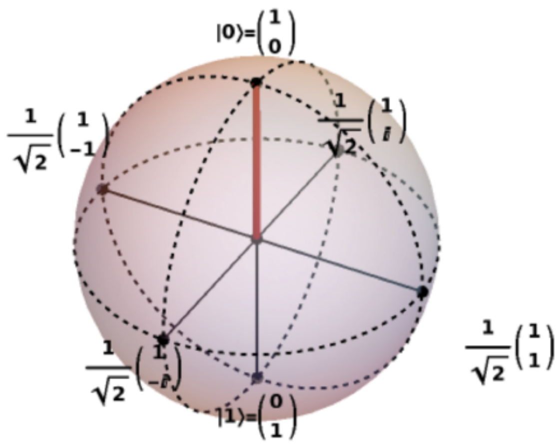

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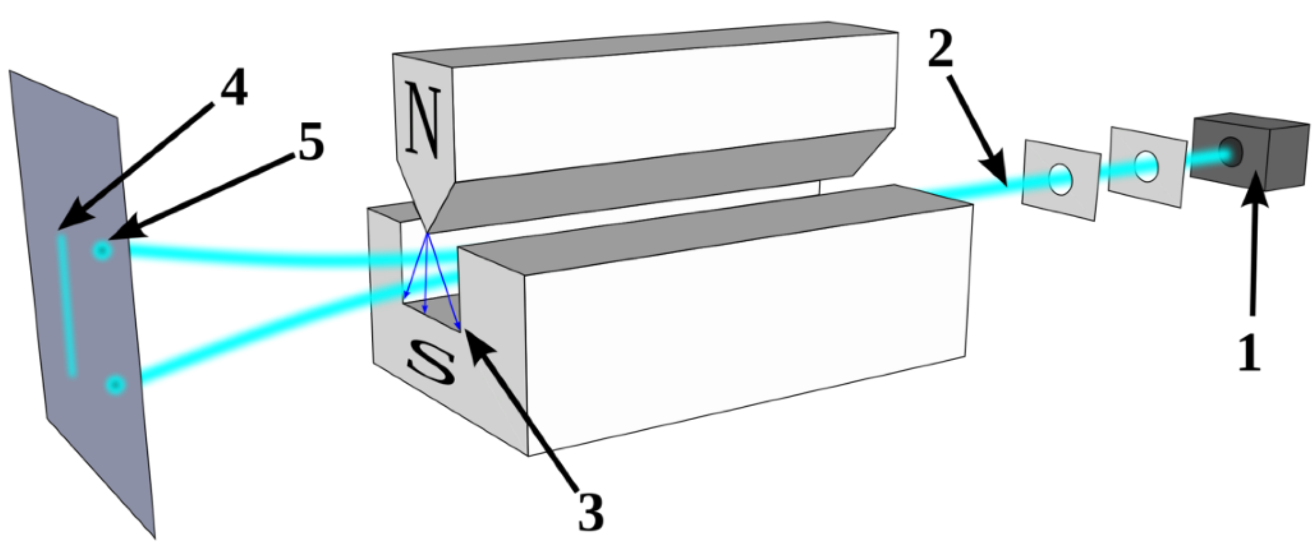
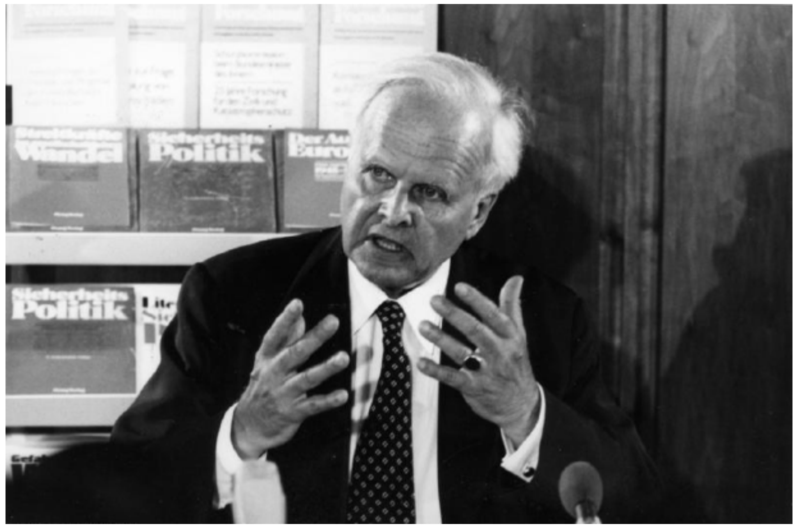
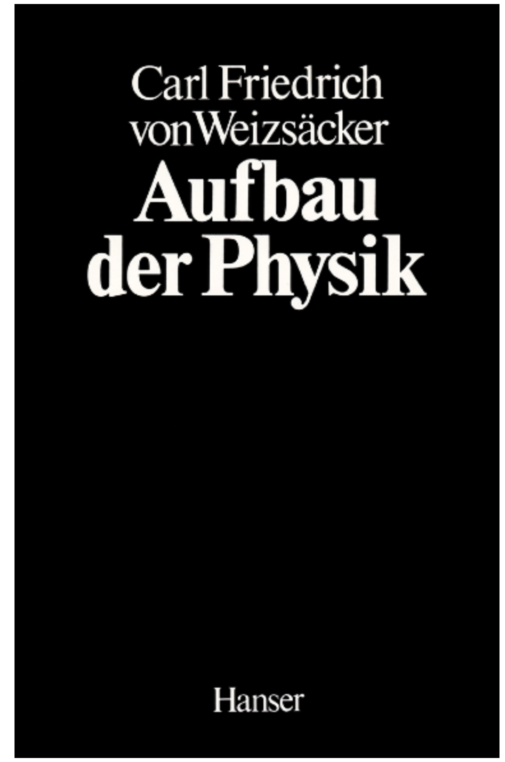


Von Weizsäcker’s theory of ”ur alternatives” (1955-58)



Carl-Friedrich von Weizsäcker
(1912-2007)



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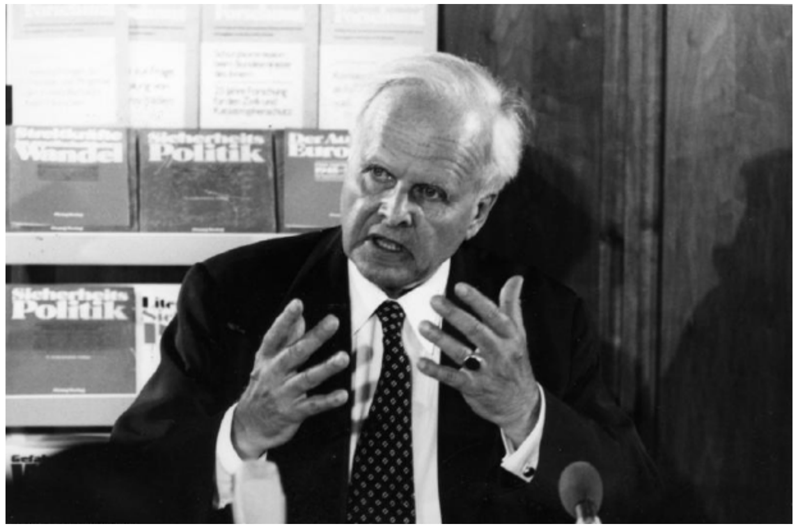

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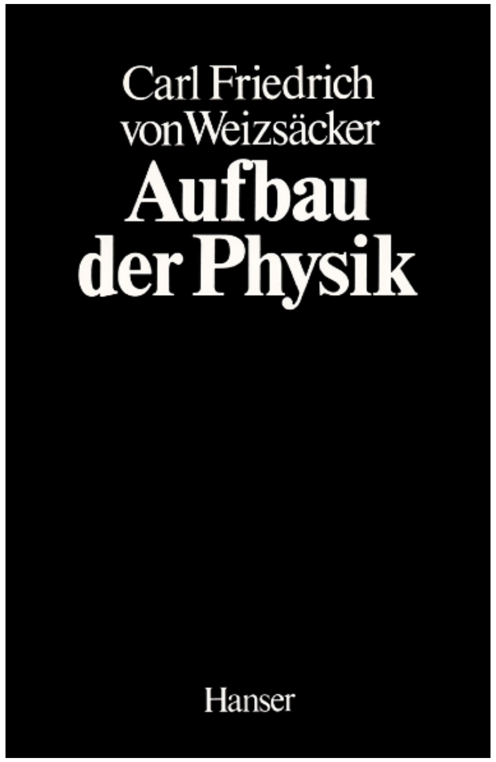

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Summary via lyre.de/urinfo.htm (imperfect English translation is mine):

Von Weizsäcker gives the following definition of the central notion of "ur-alternative":
The binary alternative, out of which the state spaces of quantum theory can be built, is called ur-alternative. The subobject associated with an ur-alternative is called "ur".

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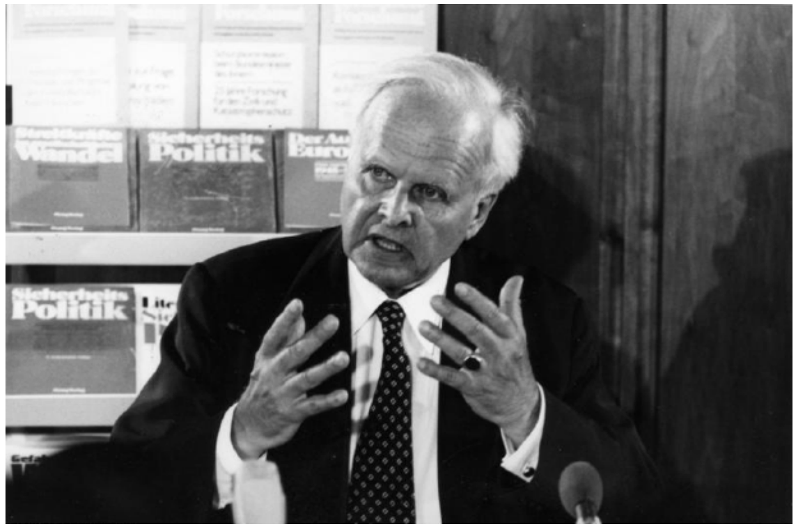

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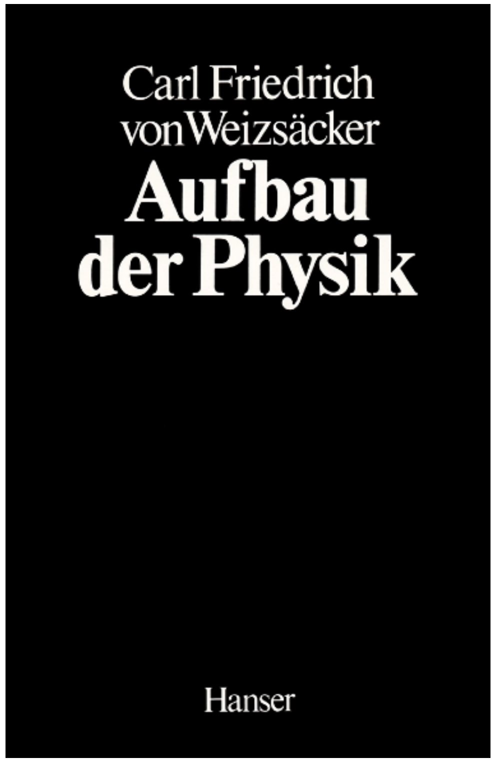

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An ur's essential symmetry group is SU(2). A world built of urs should be essentially invariant under this group. The central **fundamental assumption of ur theory** is, that space itself it a consequence of the ur-hypothesis and the symmetry group of the ur.

Bill Wootters: “statistical distance equals actual distance”

No-cloning theorem, Page-Wootters mechanism…

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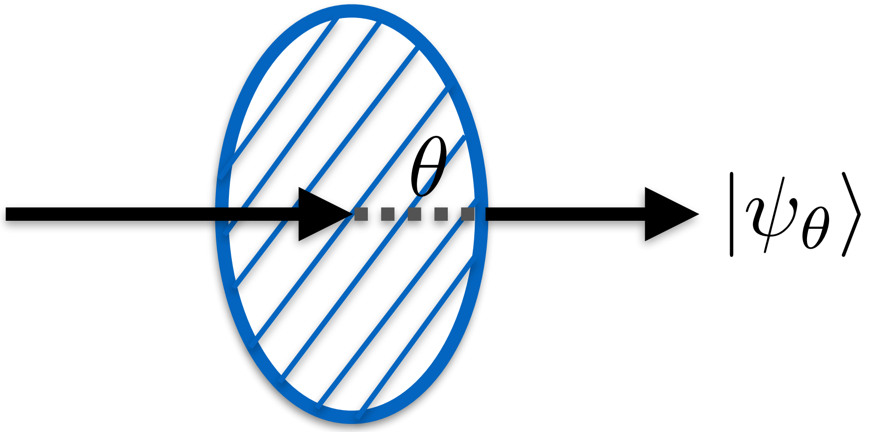

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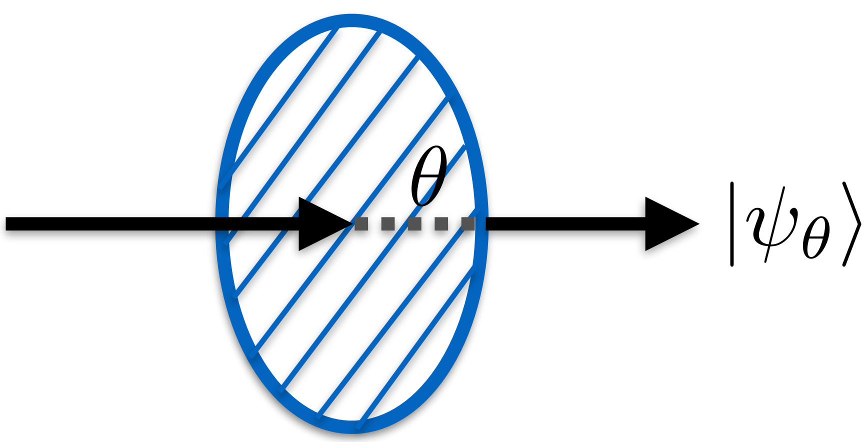

Shown **without** assuming QM:

$$\underbrace{d(\psi_\theta, \psi_{\theta'})}_{\text{statistical distance for yes-no measurement}} = c \cdot \underbrace{|\theta - \theta'|}_{\text{"actual" distance of angles}}$$

$$\Leftrightarrow p(\theta) = \cos^2 \frac{n}{2}(\theta - \theta_0)$$

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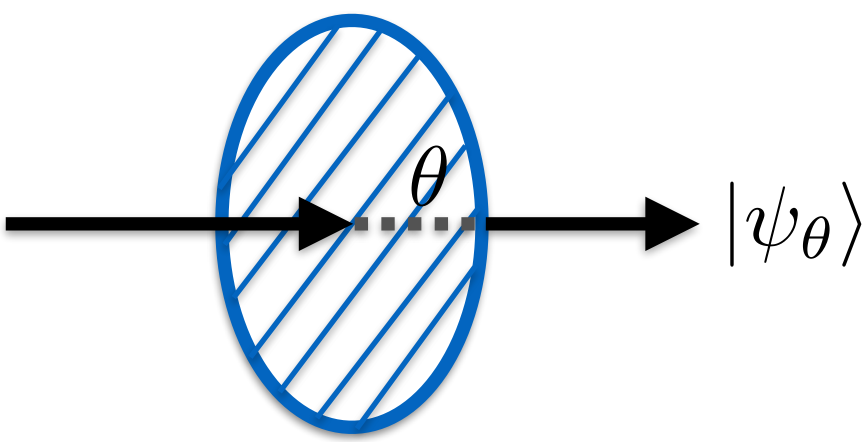

$$d(\theta_1,\theta_2)=\frac{1}{\sqrt{n}}\int_{\theta_1}^{\theta_2}\frac{d\theta}{2\Delta\theta}=\int_{\theta_1}^{\theta_2}d\theta\,\frac{|dp/d\theta|}{2[p(1-p)]^{1/2}}$$

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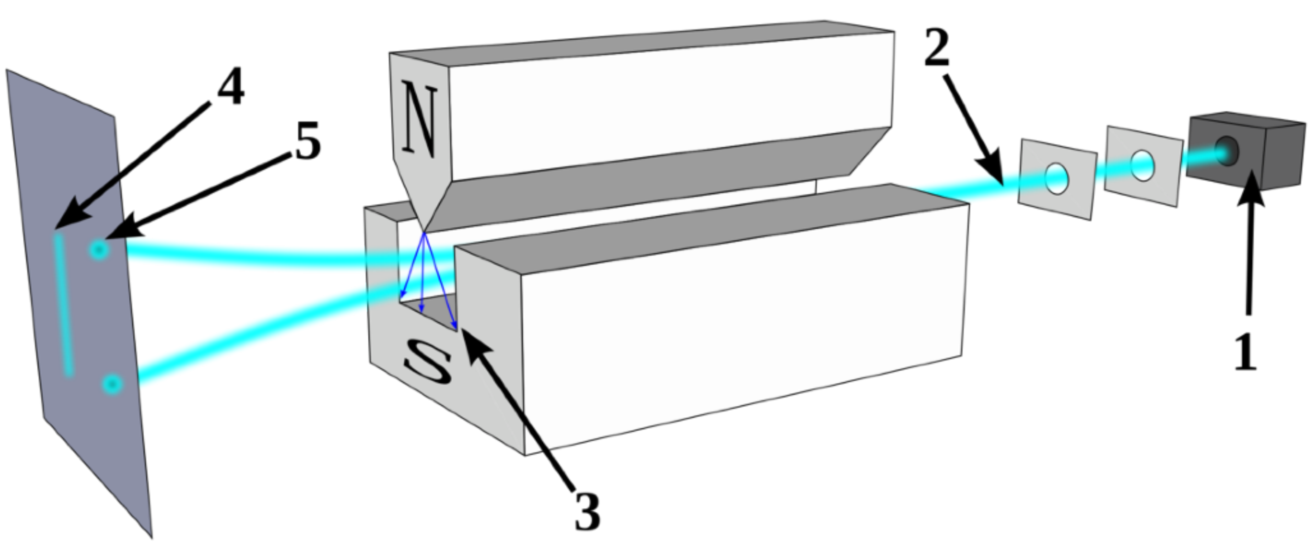

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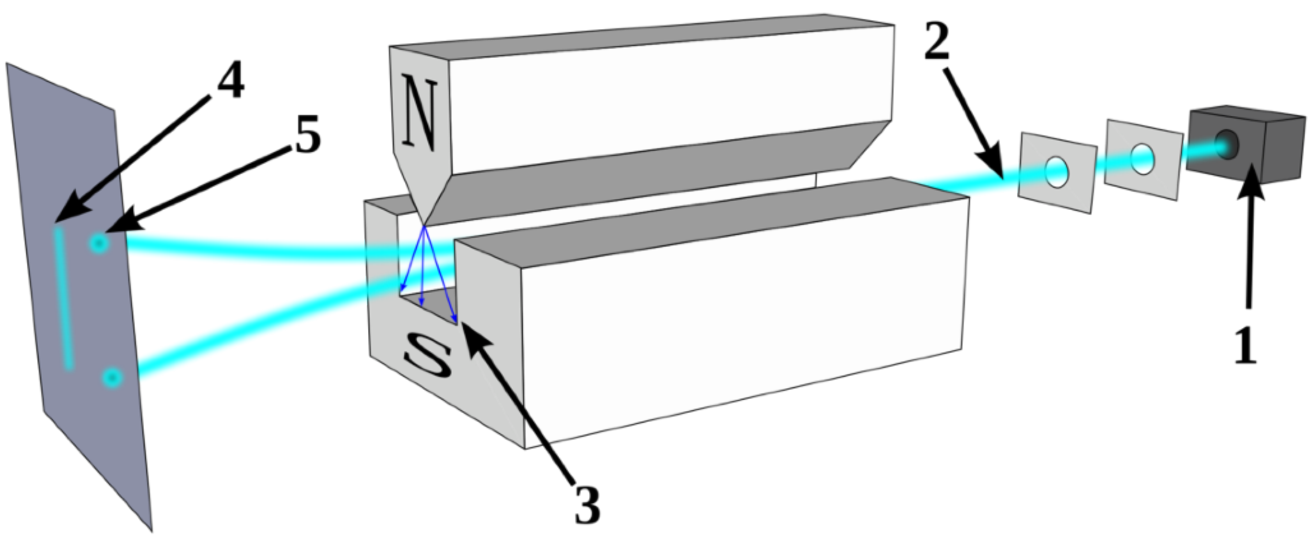
Does the structure of spacetime constrain the structure of our world's probabilistic theory, i.e. does it **imply some of the structure of QT?**



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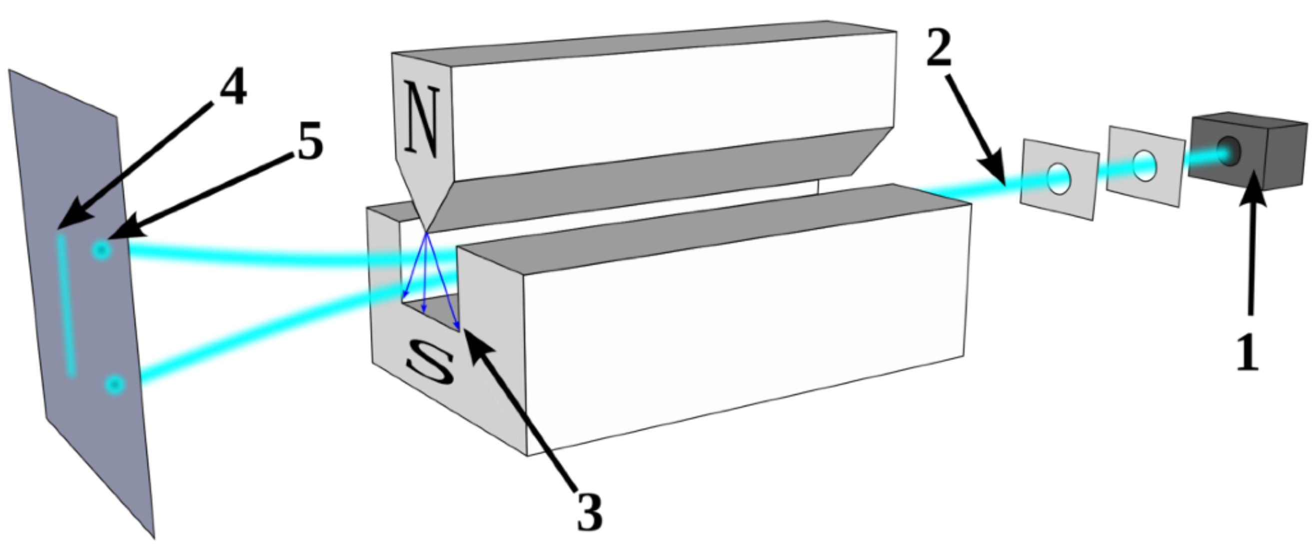

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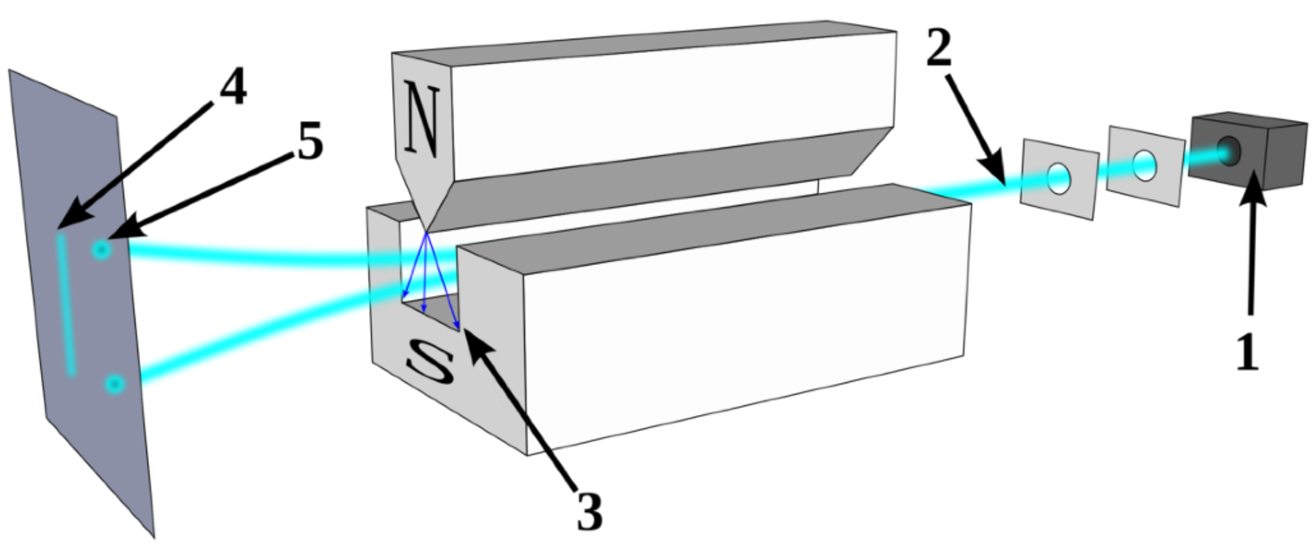
Now:
Systematic study of rotational prepare-and-measure correlations, within QT and more generally.



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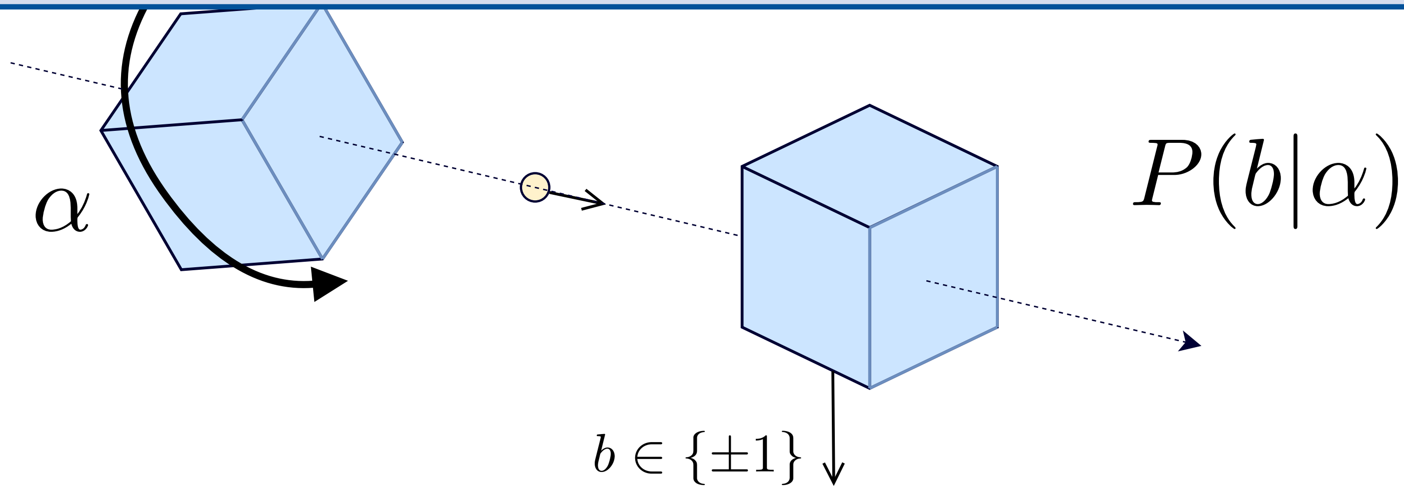

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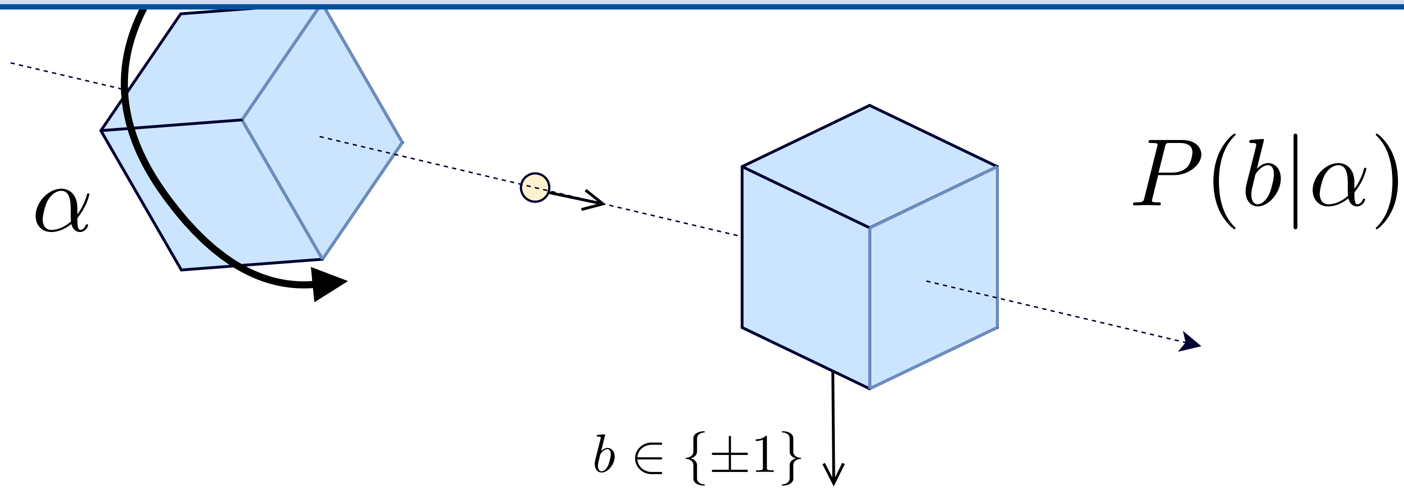

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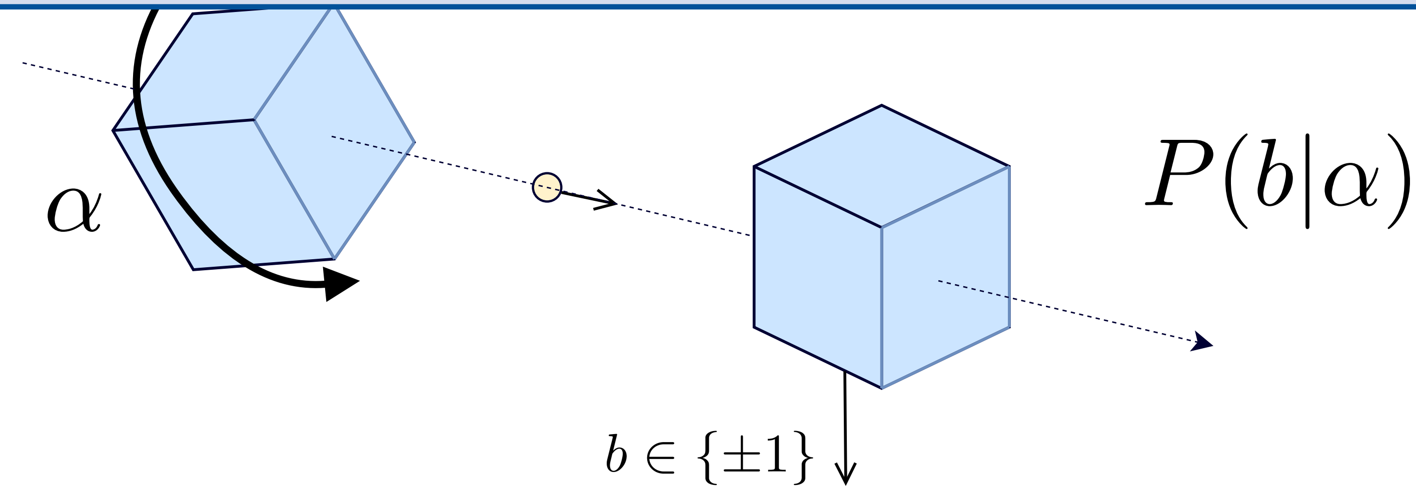

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To the space of ensembles of preparation devices, we can associate a real-linear space of possible **states** $\{\omega\}$. From standard arguments, **rotational covariance** implies that it carries a **representation** of SO(2), and

$$P(b|\alpha) = (e_b, T_\alpha \omega).$$

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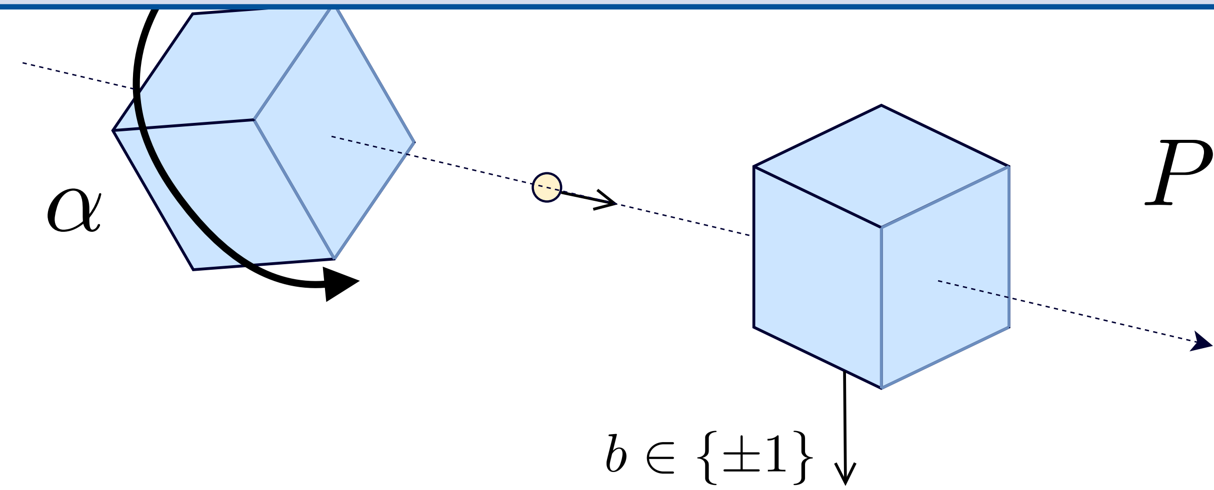

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If $\{\omega\}$ is finite-dimensional, then there is some $J \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$ (which we call the "spin" of the system) such that

$$P(b|\alpha) = c_0 + \sum_{j=1}^{2J} \left(c_j \cos(j\alpha) + s_j \sin(j\alpha) \right).$$

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Indeed, for **every** probability law

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We define the **spin-J rotation boxes** $\mathcal{R}_J$ as the set of all such $P(+1|\alpha)$, i.e. the probability-valued trigonometric polynomials of degree $\leq 2J$.

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These are exactly the probability rules arising from GPT systems that carry a representation of SO(2) where the "block of highest charge" is

$$\begin{pmatrix} \cos(2J\alpha) & -\sin(2J\alpha) \\ \sin(2J\alpha) & \cos(2J\alpha) \end{pmatrix}.$$

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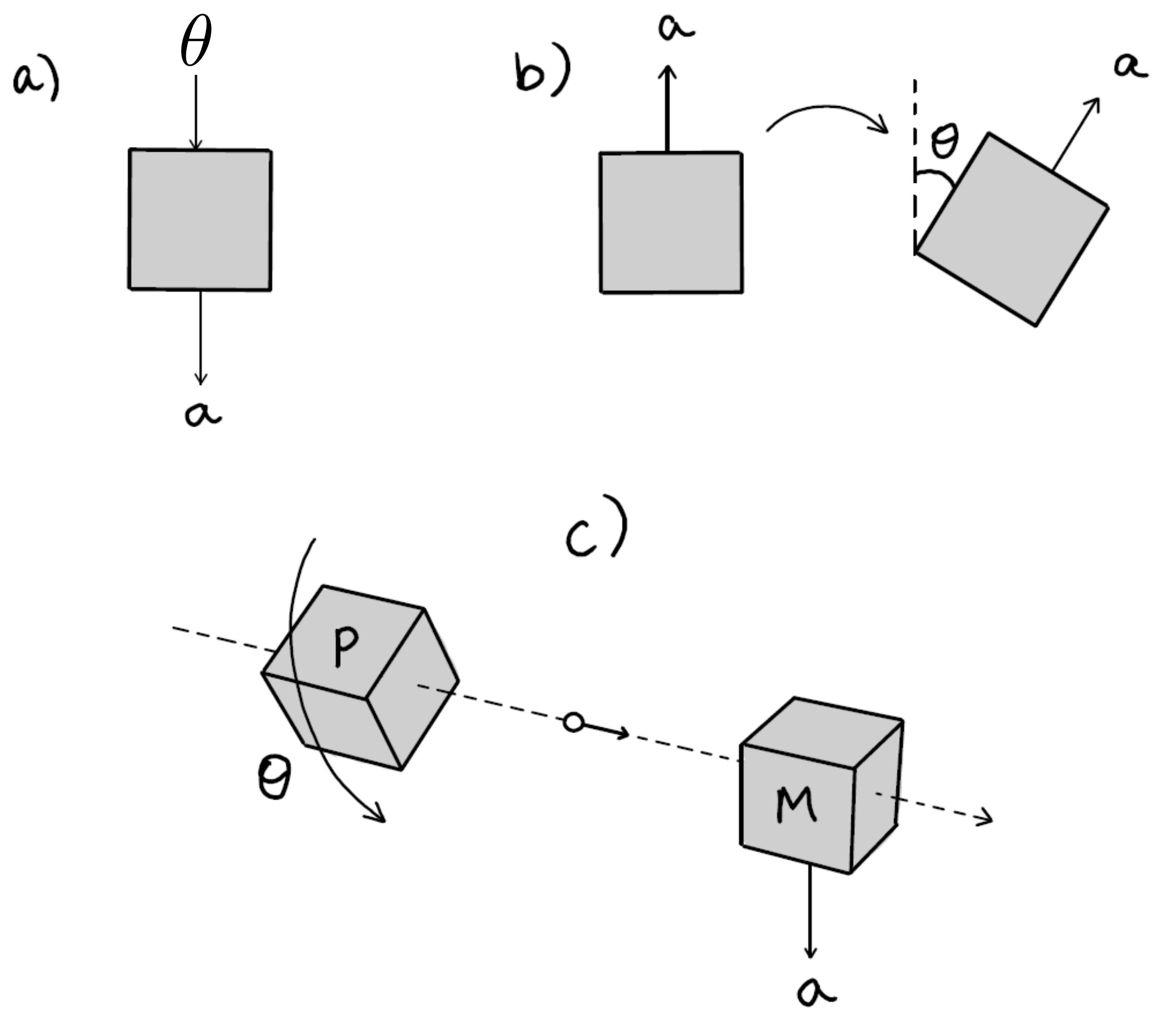

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Definition. The quantum spin-J correlations $\mathcal{Q}_J$ are the set of functions

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The sets $\mathcal{Q}_J$ are convex and compact, and they satisfy $\mathcal{Q}_J \subseteq \mathcal{R}_J$ (i.e. each function $\alpha \mapsto P(+1|\alpha) \in \mathcal{Q}_J$ is a trig. poly of degree $\leq 2J$).

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Lemma. For every $P \in \mathcal{Q}_J$, there is a pure state $|\psi\rangle \in \mathbb{C}^{2J+1}$ and a POVM $\{E_b\}_{b\in\{+1,-1\}}$ such that $P(b|\alpha) = \langle\psi|U_\alpha^\dagger E_b U_\alpha|\psi\rangle$, where $U_\alpha := \exp(i\alpha Z)$, $Z + \mathrm{diag}(J, J-1, \ldots, -J)$.

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For $J = 0$, we obtain the constant probability functions:

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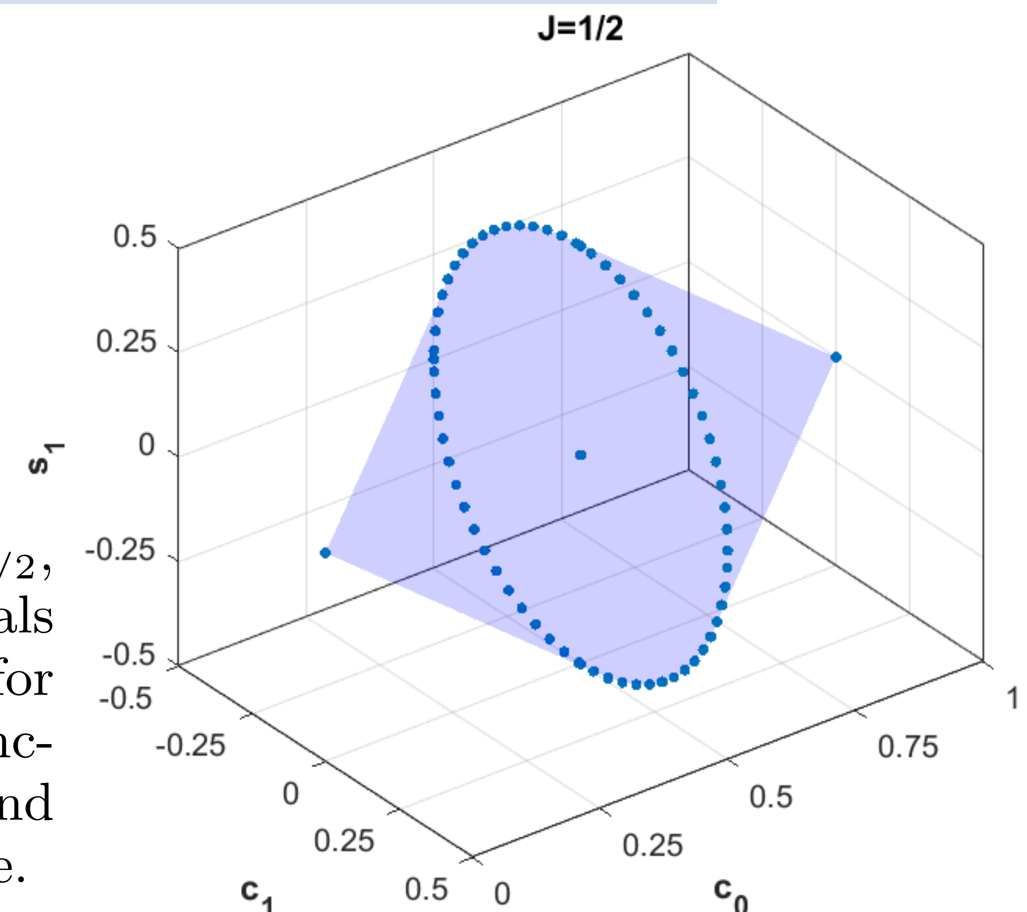

FIG. 4. The binary quantum spin-1/2 correlations $\mathcal{Q}_{1/2}$, which happens to be the set of trigonometric polynomials $P(+|\theta) = c_0 + c_1 \cos\theta + s_1 \sin\theta$ with $0 \leq P(+|\theta) \leq 1$ for all θ. The two endpoints are the constant zero and one functions, and the other extremal points on the circle correspond to functions $\theta \mapsto \frac{1}{2} + \frac{1}{2}\cos(\theta - \varphi)$, with φ some fixed angle.

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$$0 \leq c_0 + c_1 \cos\alpha + s_1 \sin\alpha + c_2 \cos(2\alpha) + s_2 \sin(2\alpha) \leq 1,$$

with coefficients $(c_0, c_1, s_1, c_2, s_2) \in \mathbb{R}^5$.

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$$0 \leq c_0 + c_1 \cos\alpha + s_1 \sin\alpha + c_2 \cos(2\alpha) + s_2 \sin(2\alpha) \leq 1,$$

with coefficients $(c_0, c_1, s_1, c_2, s_2) \in \mathbb{R}^5$.

Task: classify the **extremal points** $p(\alpha)$ of the compact convex set $\mathcal{R}_1$.

Each one attains the value zero somewhere. Shifting the angle, we can restrict our attention to those with $p(0) = 0 \Rightarrow p'(0) = 0$.

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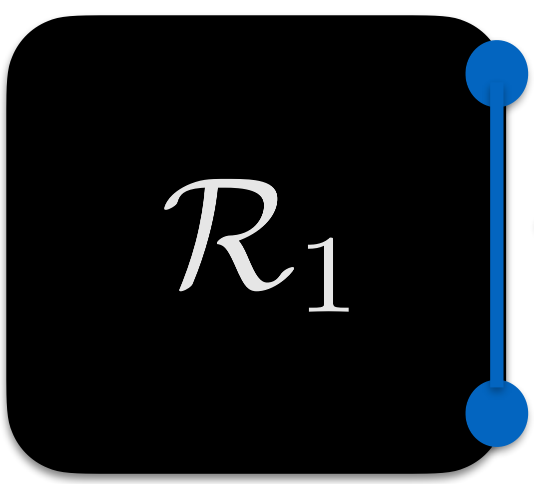
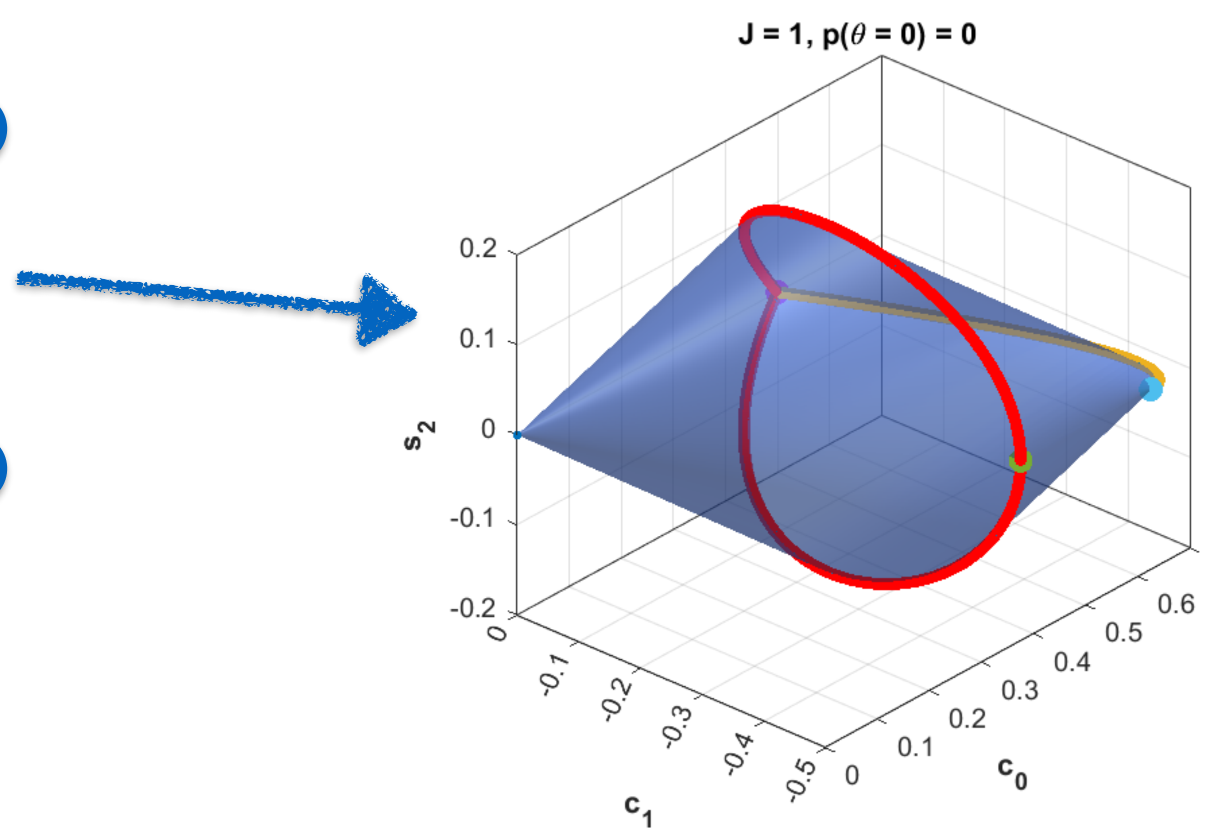


5-dimensional



3-dimensional face
$\{p \in \mathcal{R}_1 \mid p(0) = 0\}$

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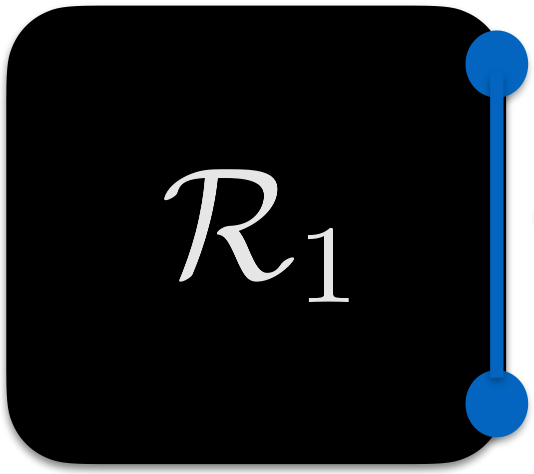

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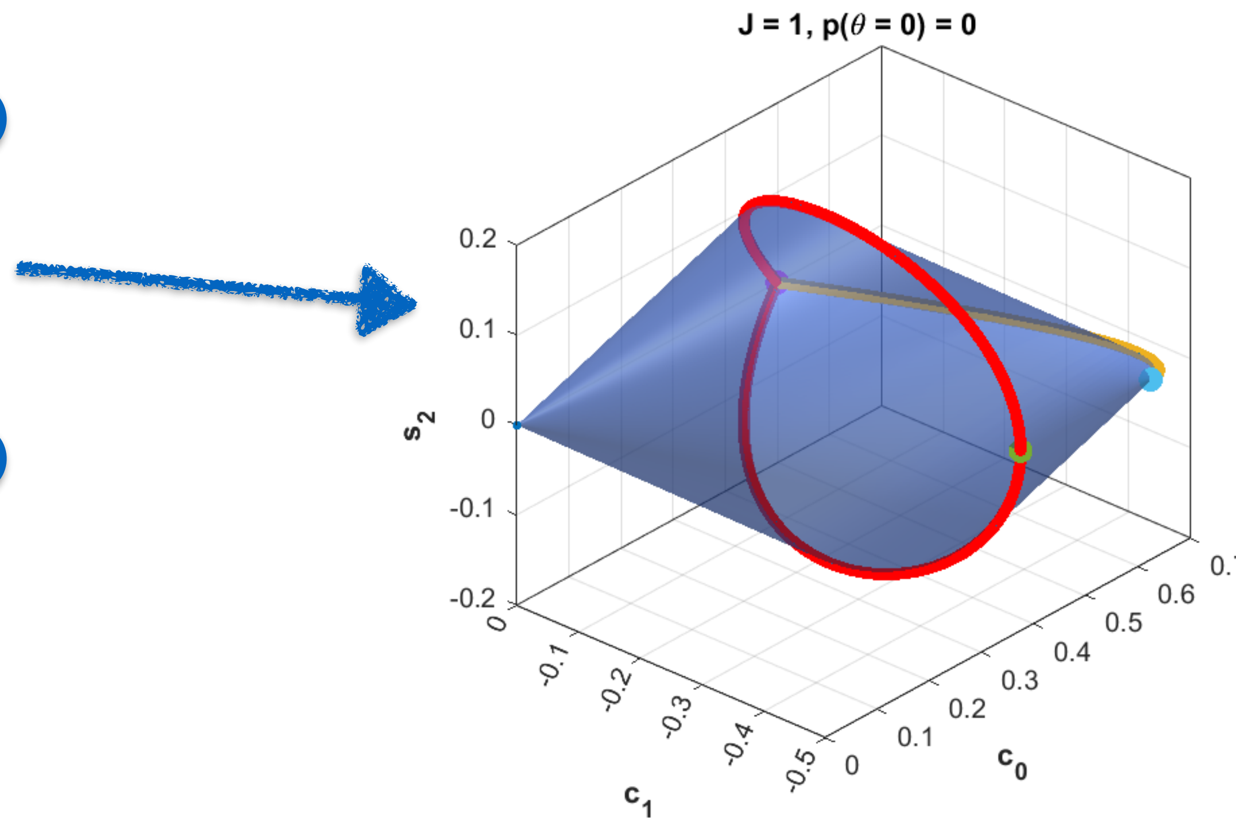

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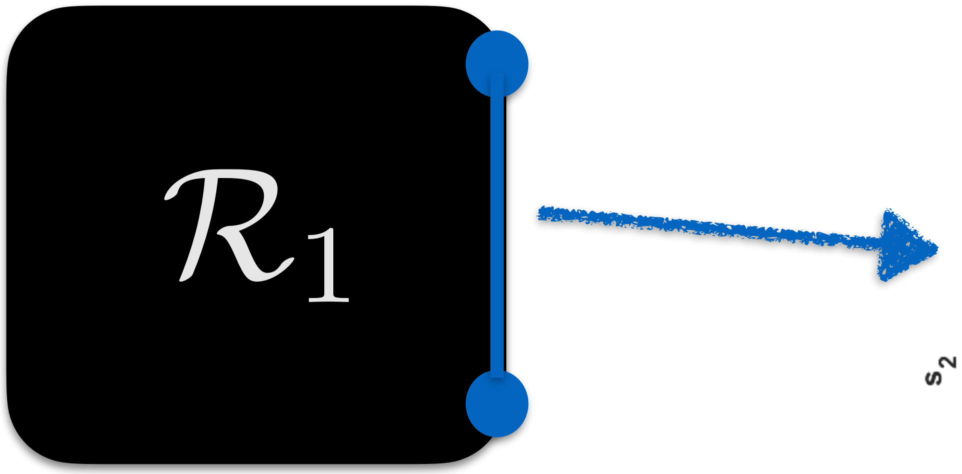

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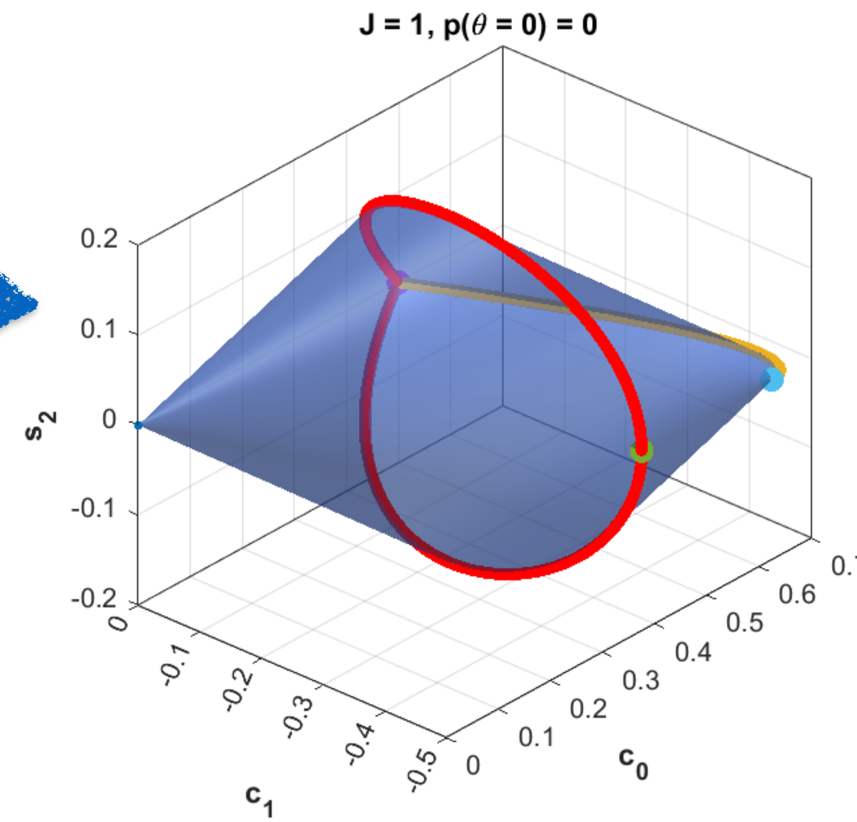

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We can give the extremal points explicitly:

$$\begin{aligned}
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Main math. tool: Fejér-Riesz theorem.

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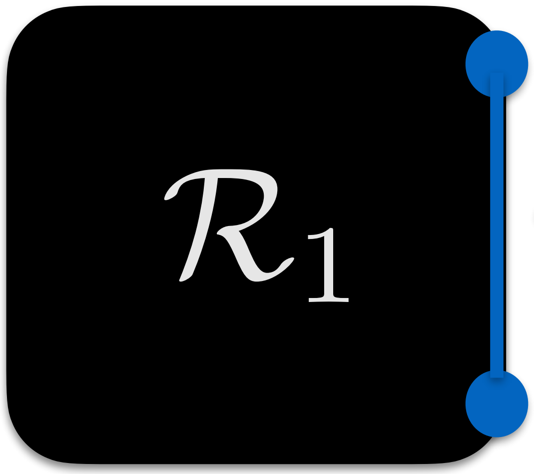

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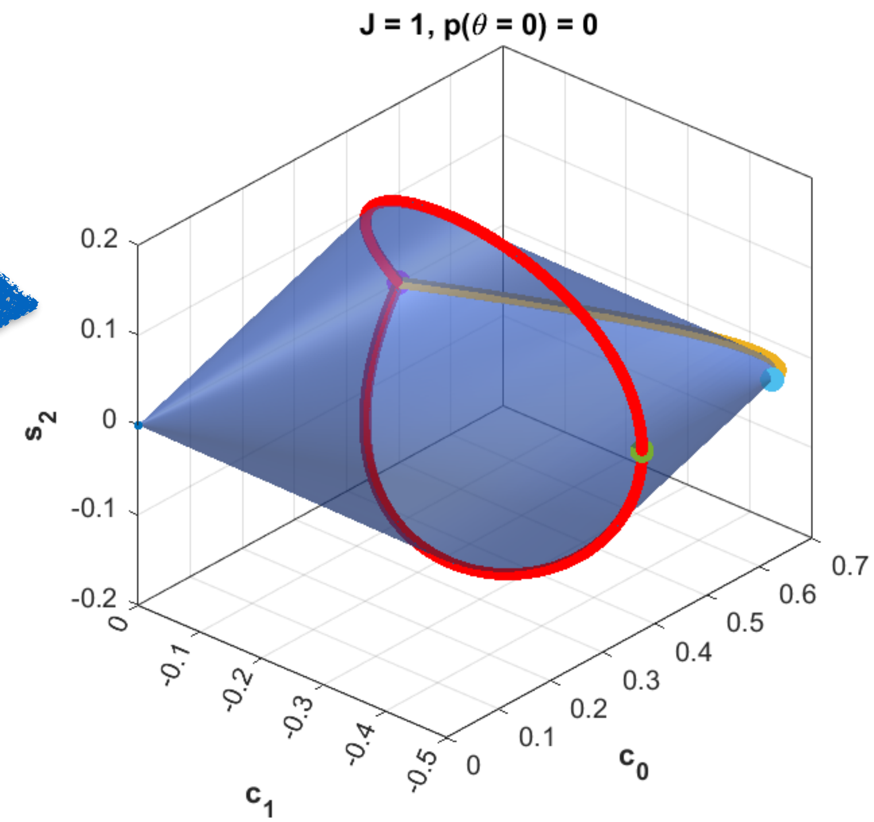

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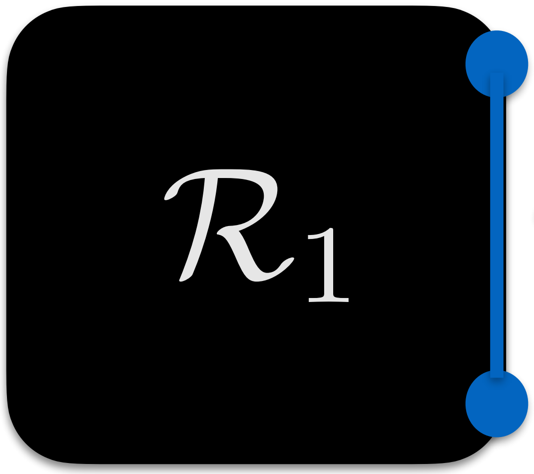

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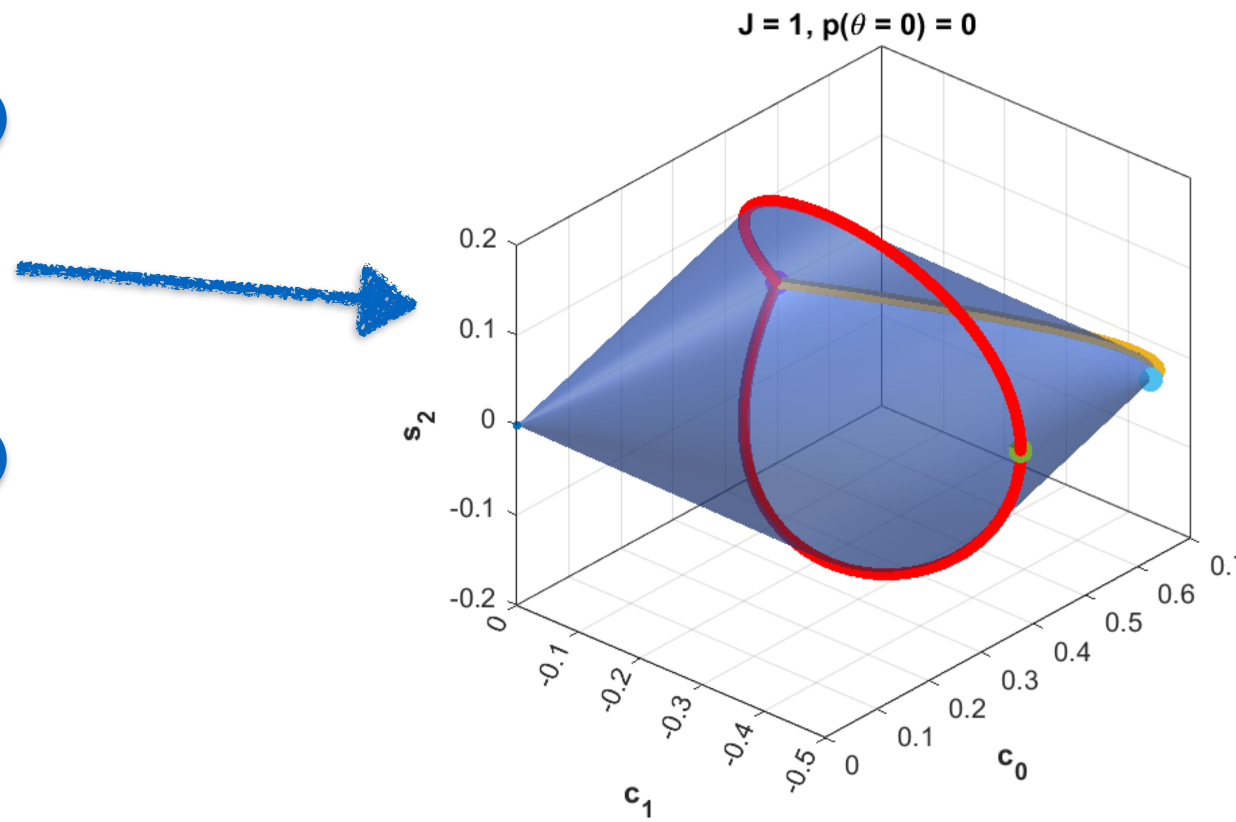

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What about $J \geq 3/2$? Soon…

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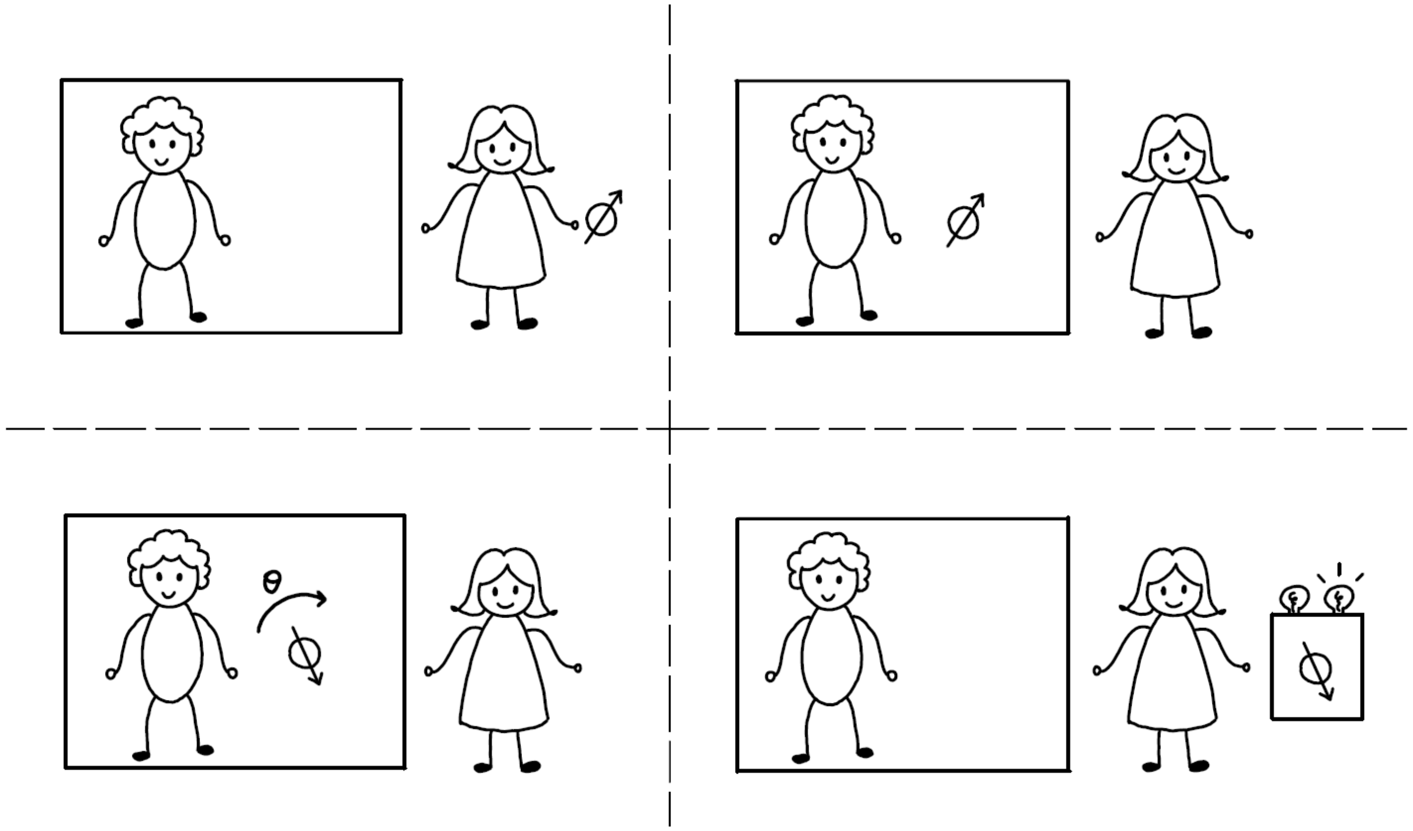

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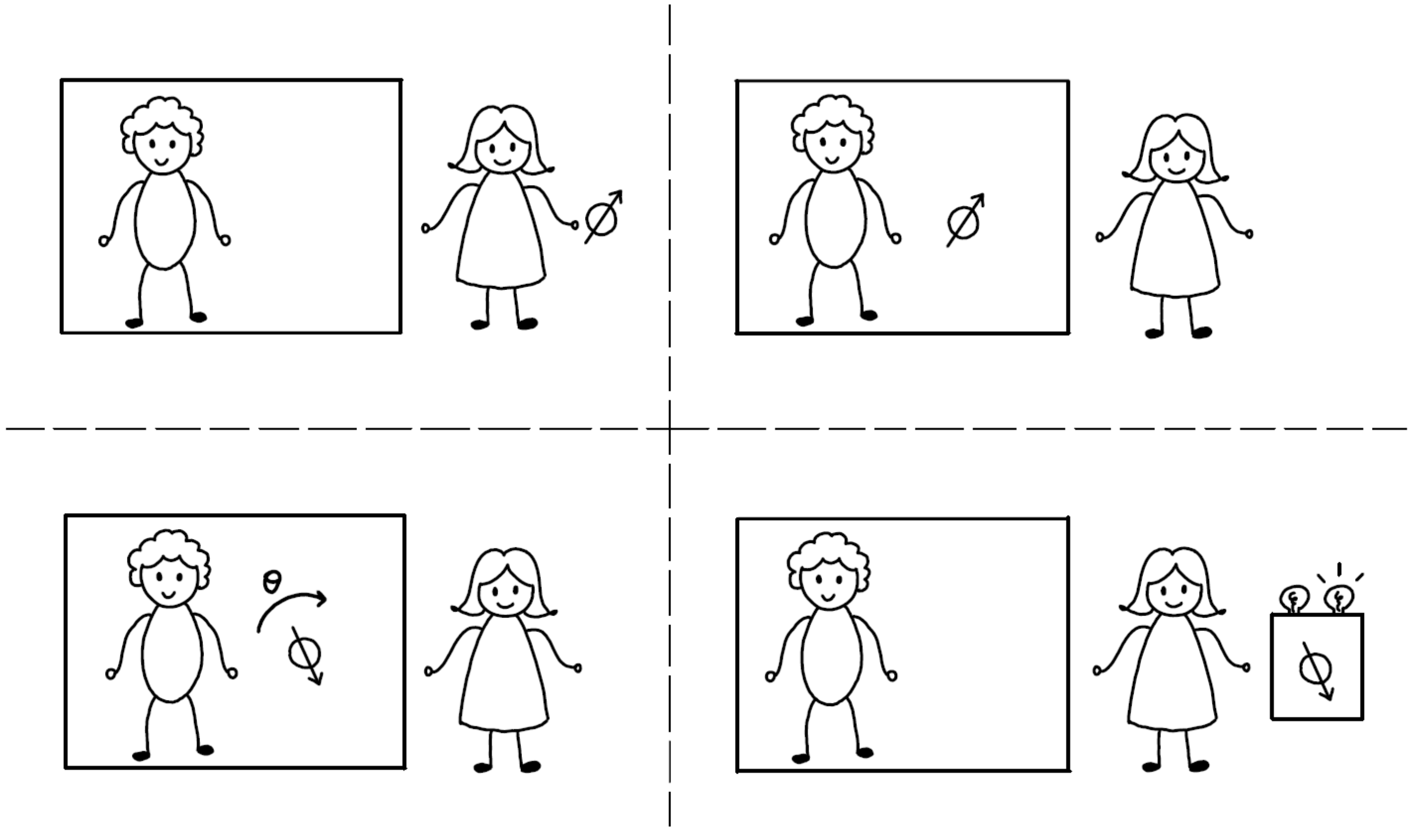

- Alice prepares a spin-J system of her probabilistic theory.
- She hands it over to Bob, who performs a rotation by $\theta.$ His angle is chosen at random with $\mu(\theta) := n^{-1}|\cos(2\theta) + \sin(3\theta)|.$

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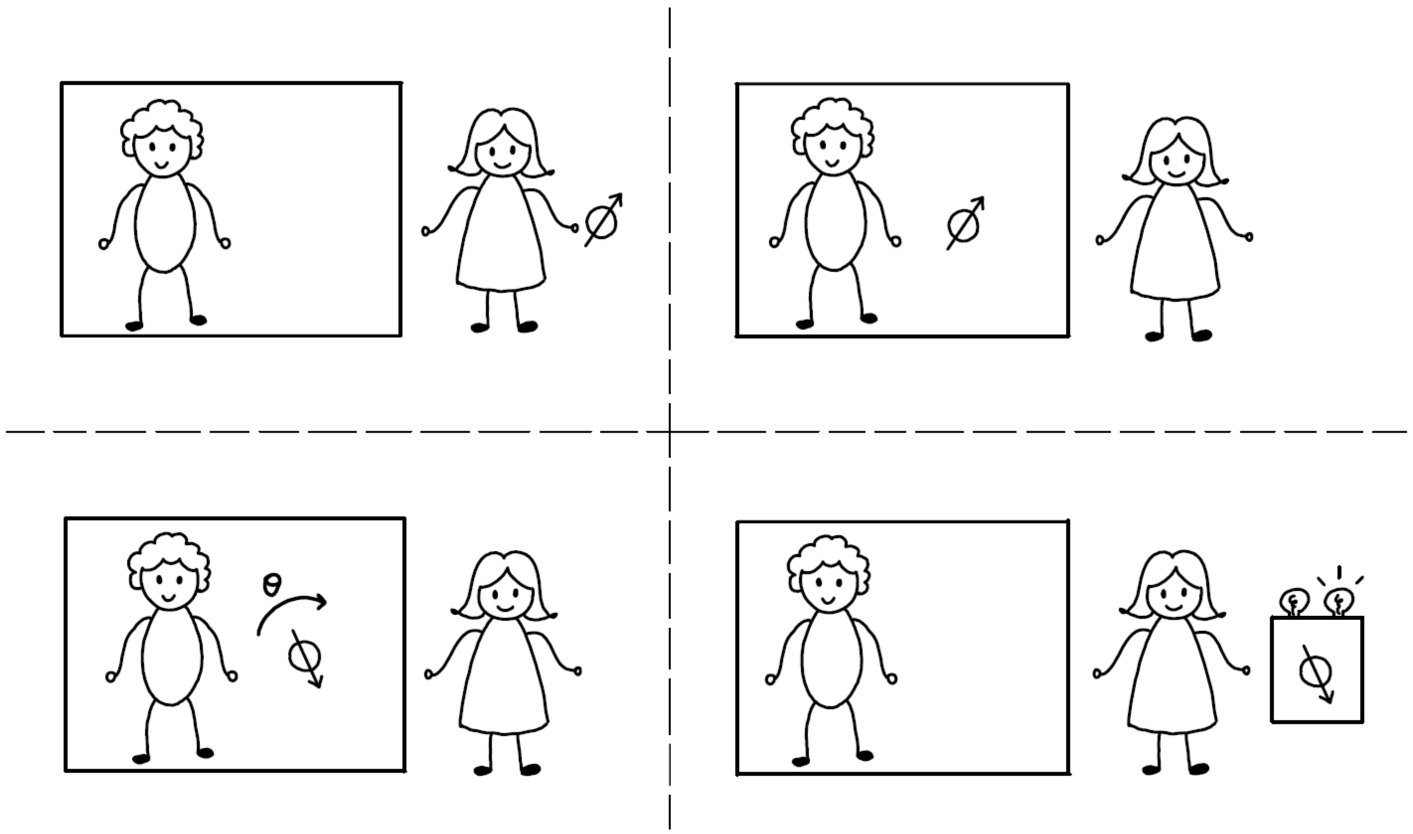

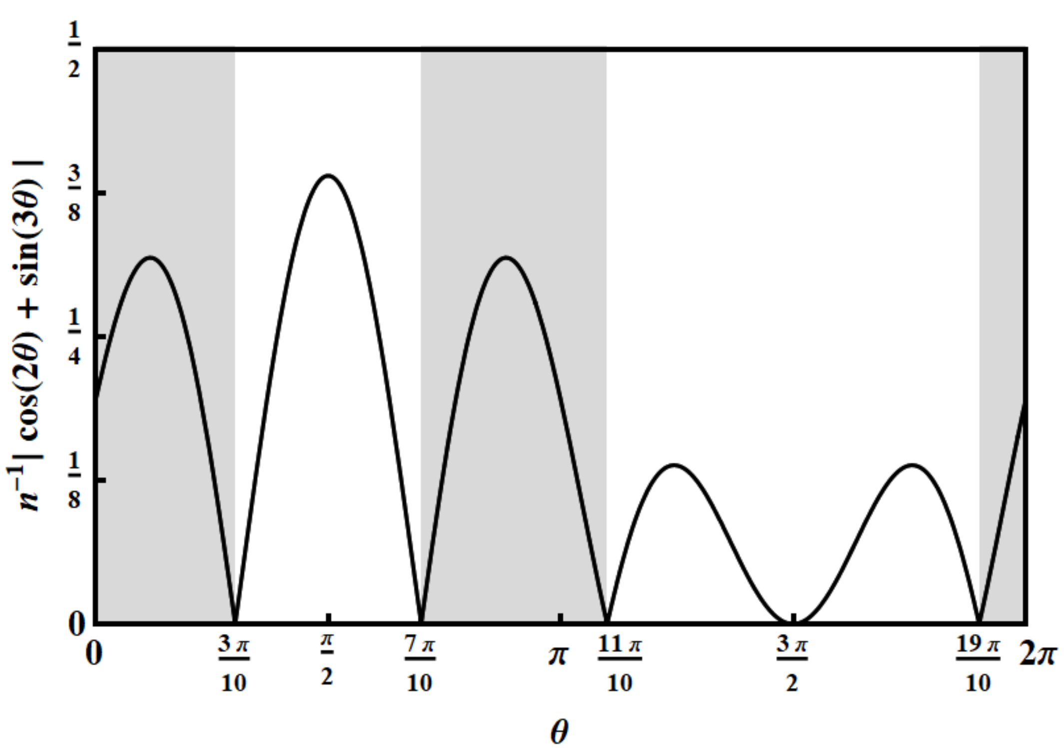

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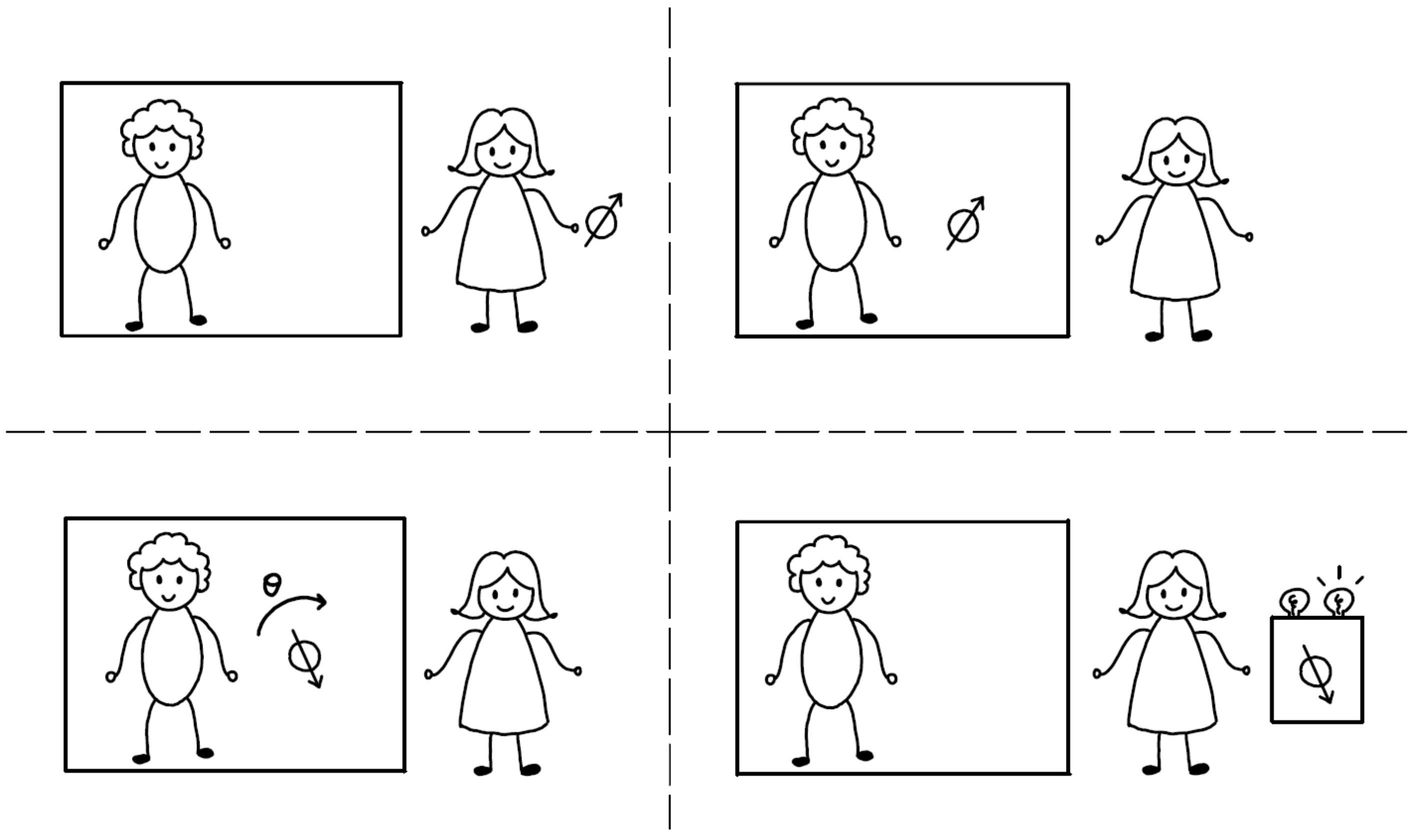

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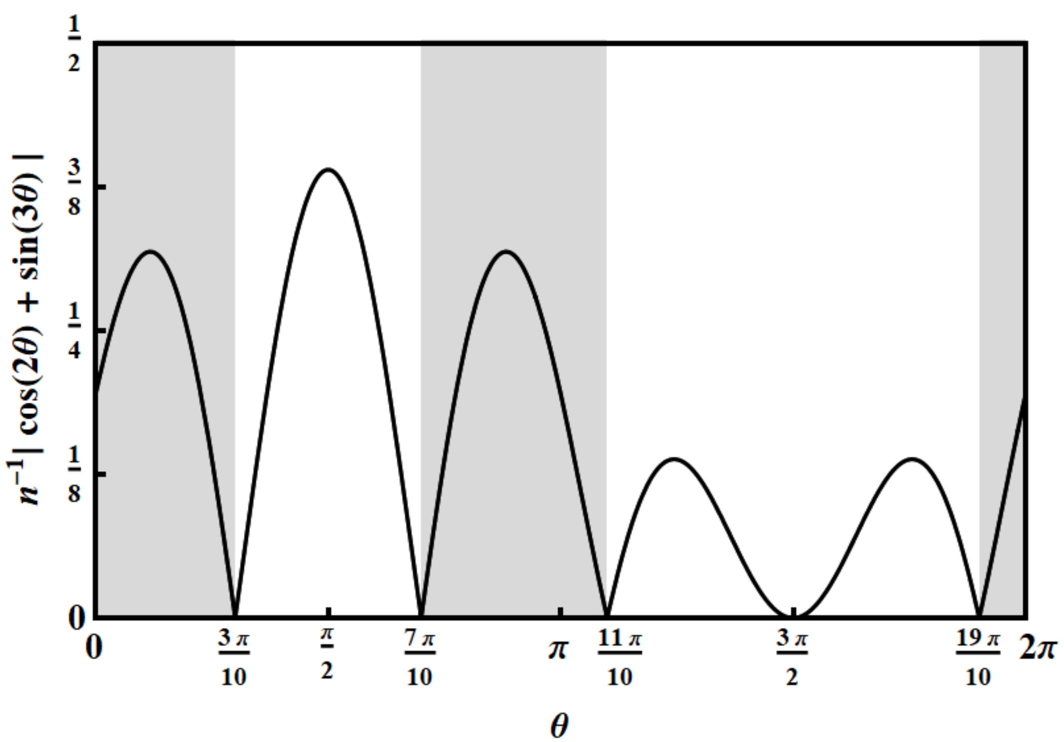

For spin $J \in \{0, \frac{1}{2}, 1\}$, quantum systems are optimal for winning such games. But for $J = \frac{3}{2}$, the maximal quantum winning probability is $P_{\text{succ}}^{\text{Q}} \approx 85\%$, whereas some rotation boxes allow Alice to win it with $P_{\text{succ}}^{\text{R}} \approx 88\%$.

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Theorem 6. *If $p \in \mathcal{Q}_{3/2}$, then its trigonometric coefficients, as taken from representation (31), satisfy*

$$c_2 + s_3 \leq \frac{1}{\sqrt{3}} \lesssim 0.5774.$$

On the other hand, the trigonometric polynomial

$$p^\star(\theta) := \frac{2}{5} + \frac{1}{4}\sin\theta + \frac{7}{20}\cos(2\theta) + \frac{1}{4}\sin(3\theta)$$

satisfies $0 \leq p^\star(\theta) \leq 1$ for all θ, hence $p^\star \in \mathcal{R}_{3/2}$, but $c_2 + s_3 = 0.6$, i.e. $p^\star \notin \mathcal{Q}_{3/2}$. In particular, $\mathcal{Q}_{3/2} \subsetneq \mathcal{R}_{3/2}$.

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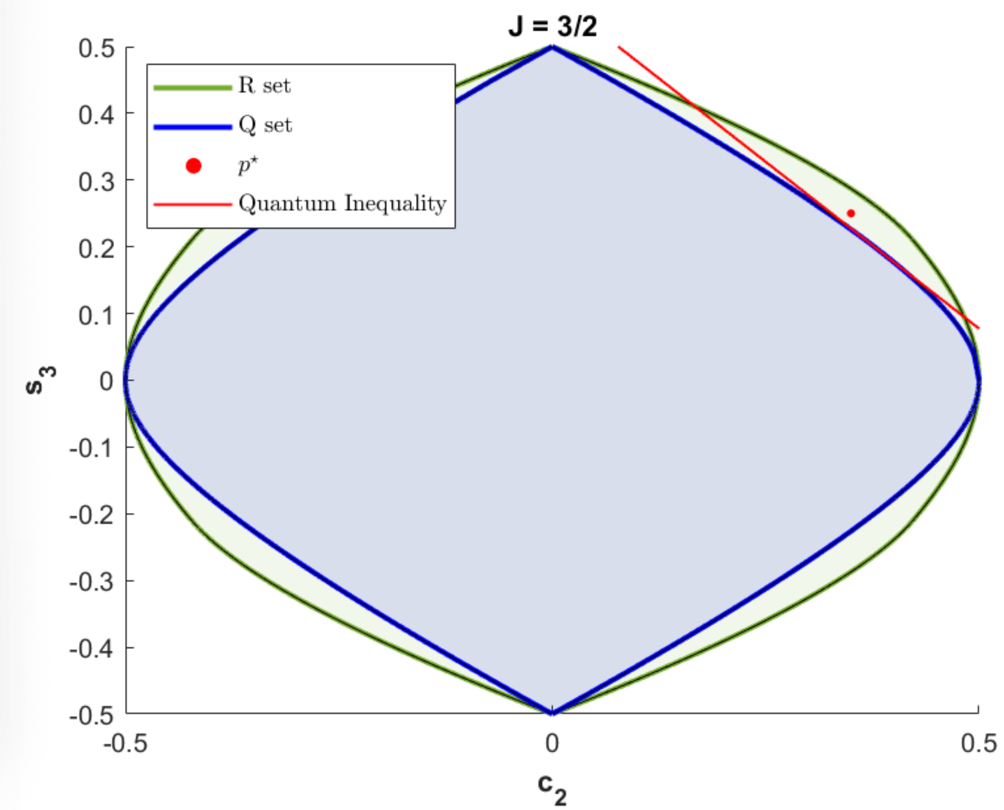

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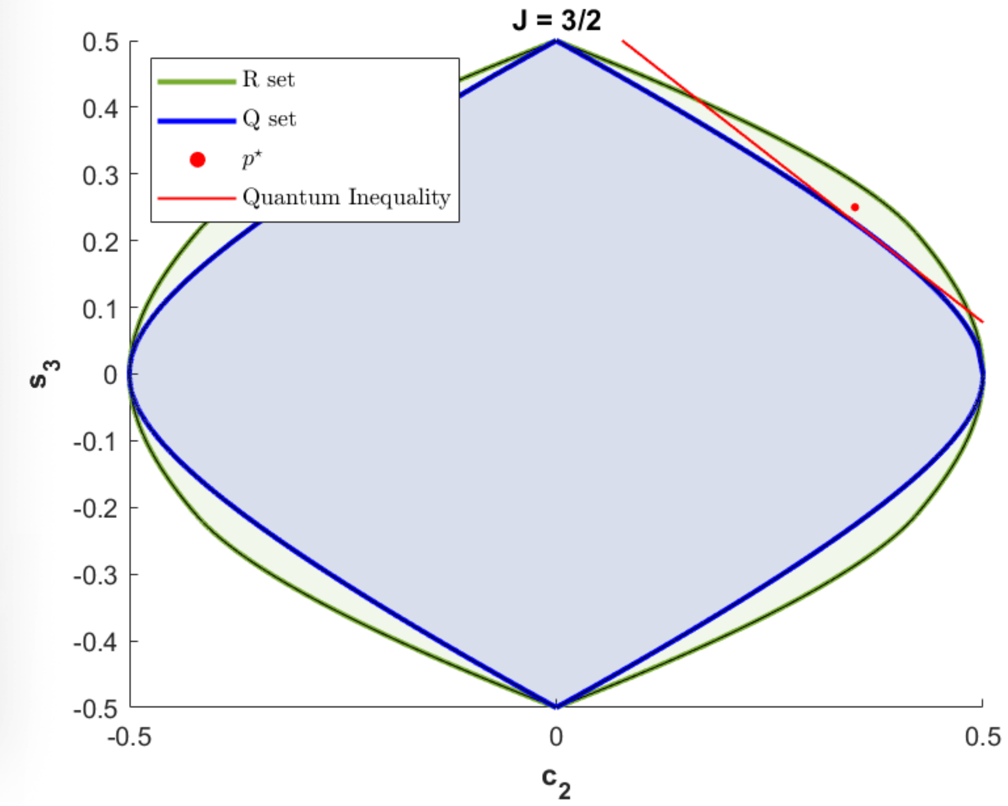

A similar gap can be demonstrated for all $J \geq 2$.

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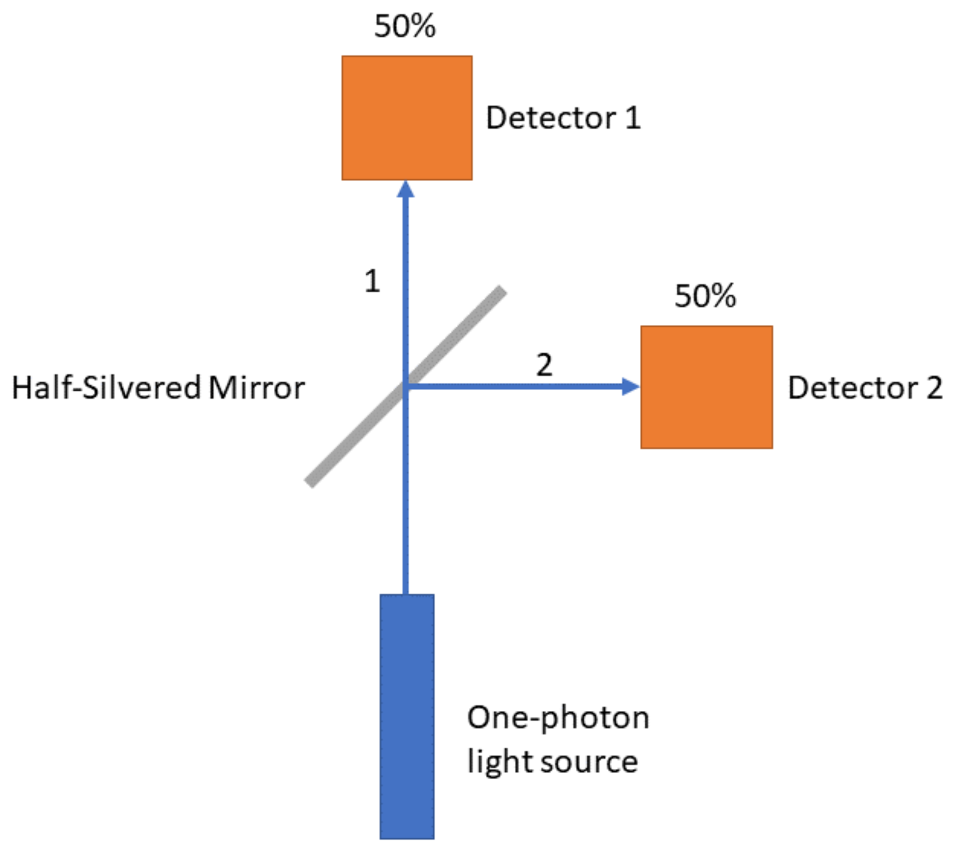

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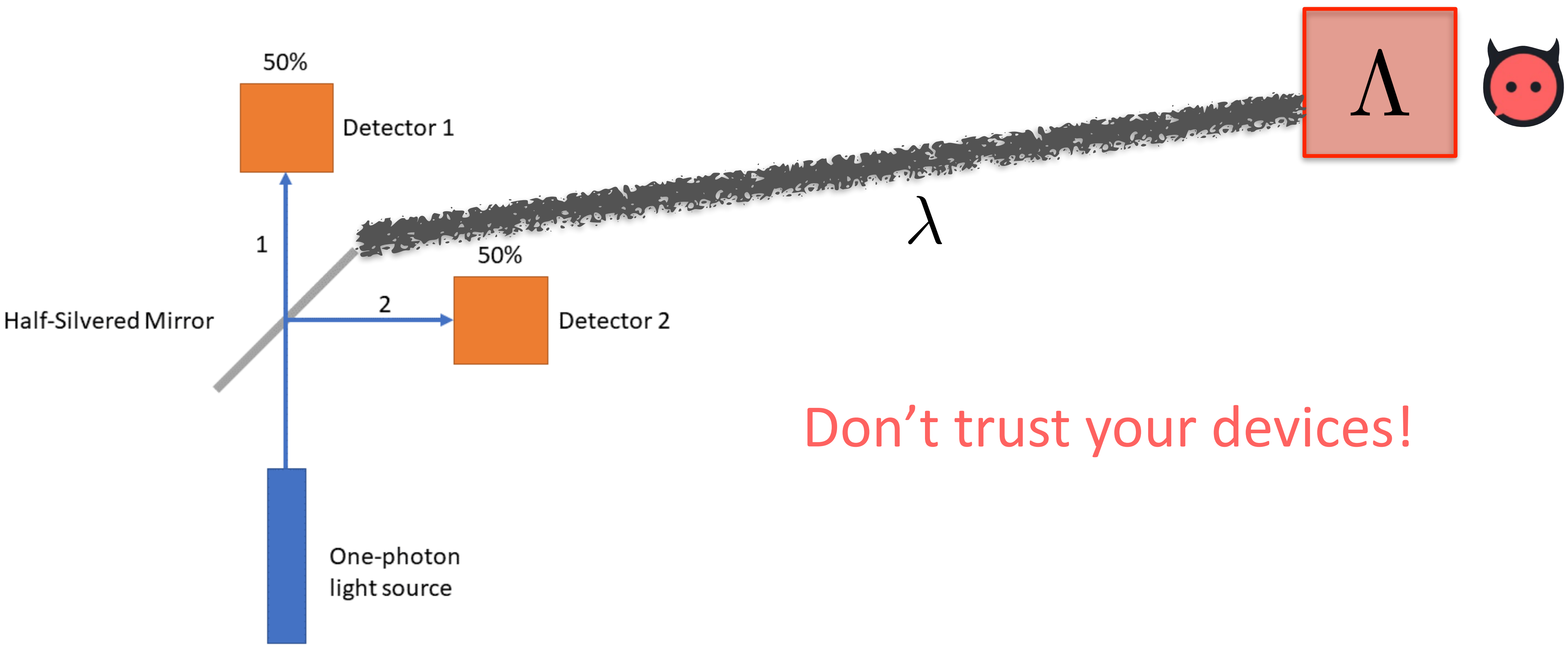

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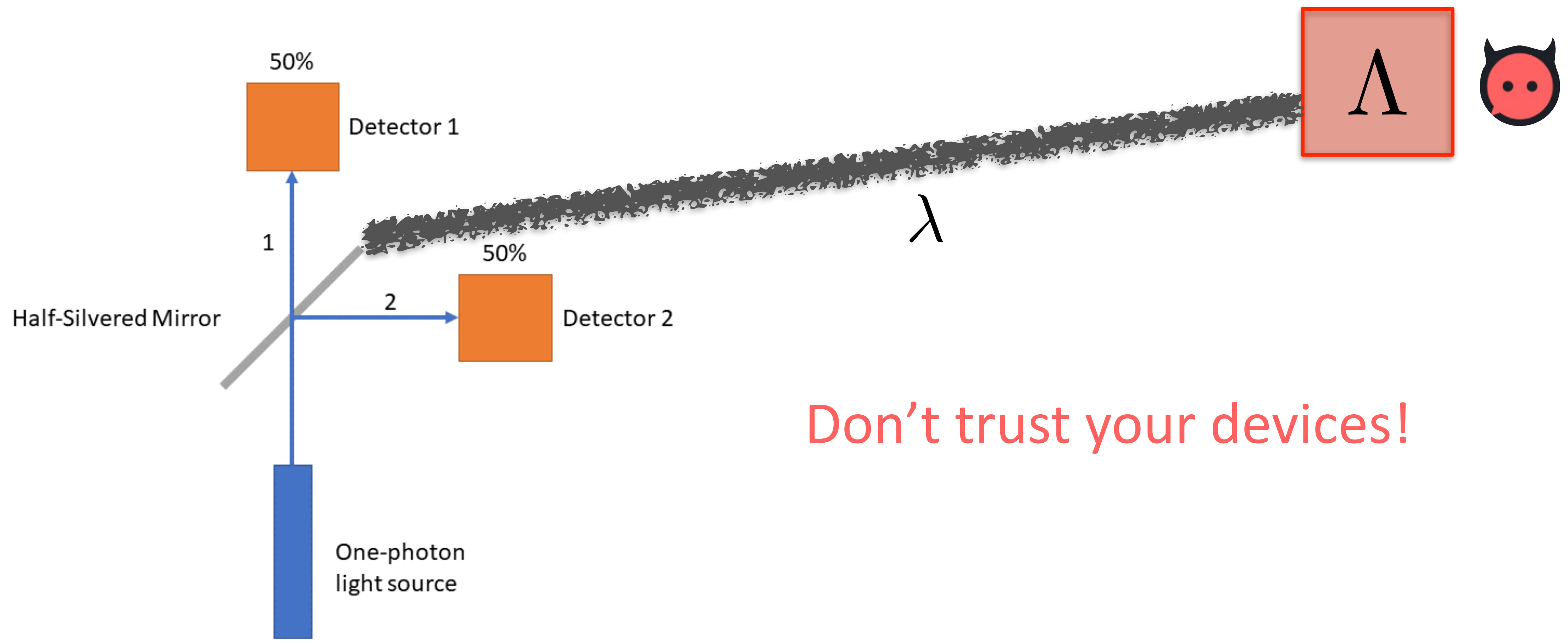

Don't trust your devices!

Device-independent randomness expansion:

Violation of Bell inequality $\Rightarrow$ outcomes uncorrelated with rest of the world

See e.g.: A. Acín, *Randomness and quantum non-locality*, QCRYPT 2012 talk.
V. Scarani, *Bell nonlocality*, Oxford Graduate Texts (2019).

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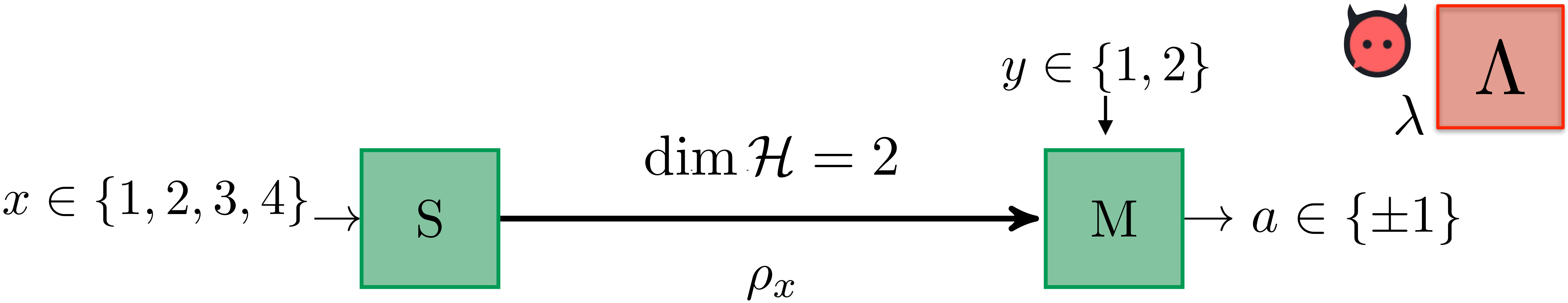

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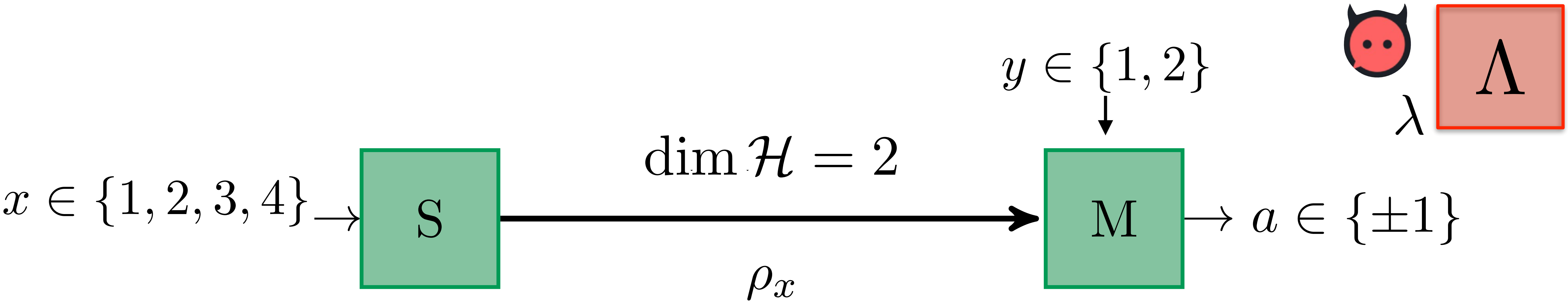

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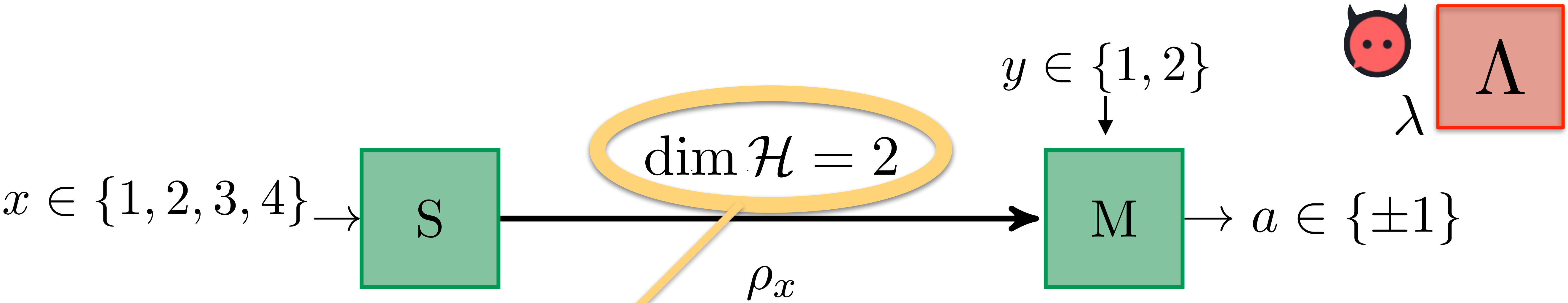

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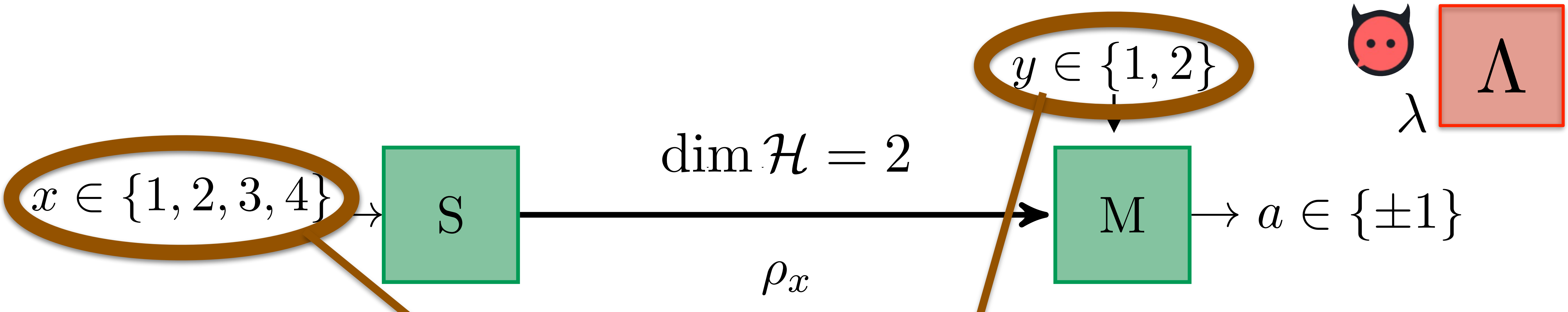

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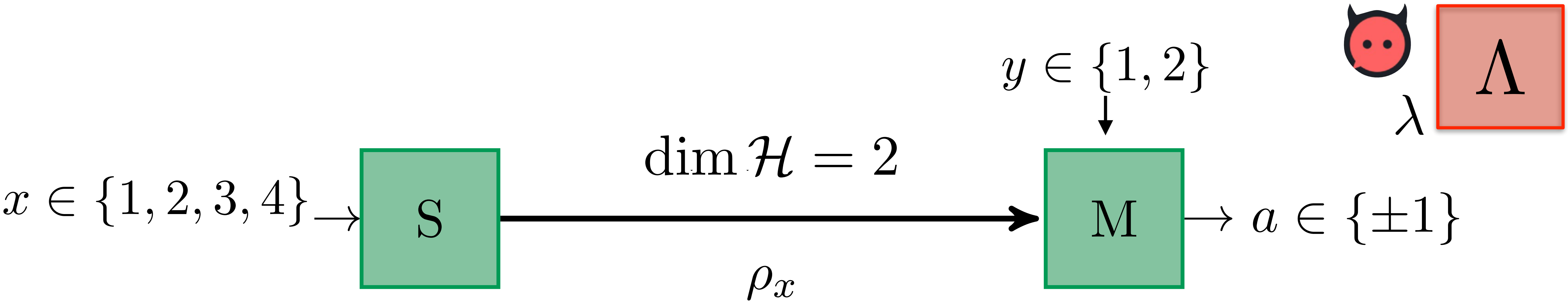

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Idea: use the formalism of rotation boxes; replace dim bound by spin bound.
Slightly more physical; does not assume the validity of quantum theory.

A theory-independent SDI randomness generator

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$$\mathcal{Q}_{J,\alpha} = \{(E_1, E_2) \mid P \in \mathcal{Q}_J\}$$
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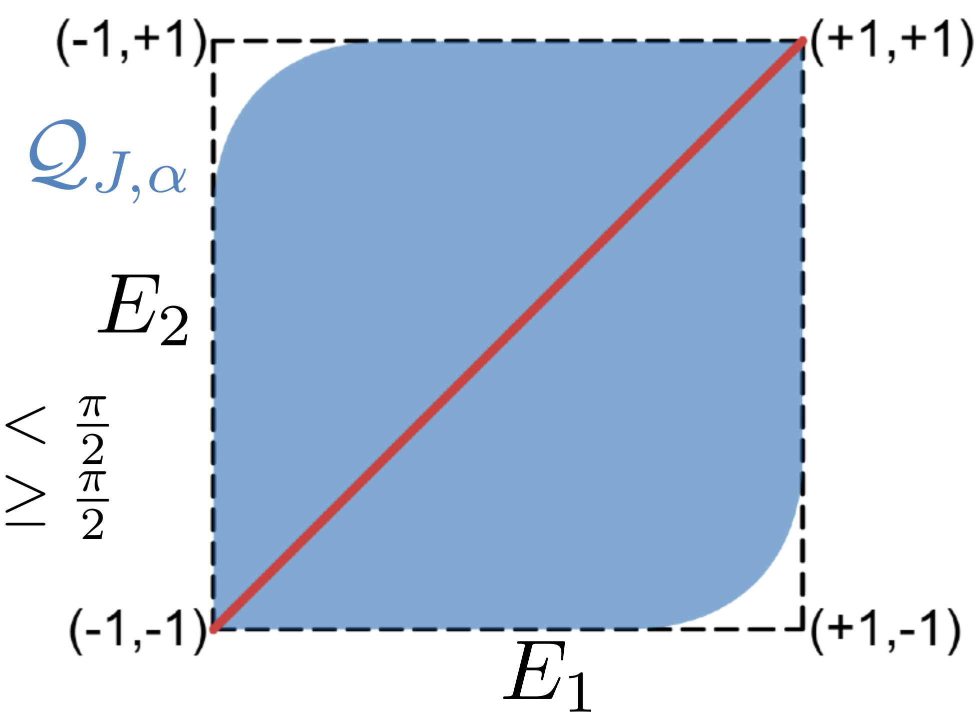

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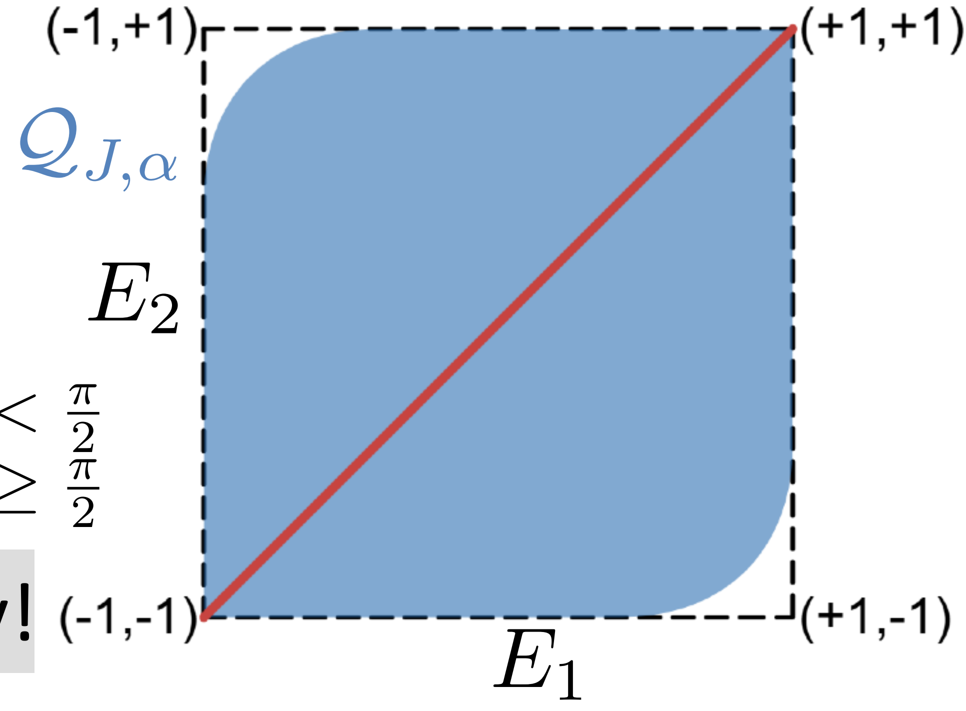

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Correlations characterized by rotational symmetry!

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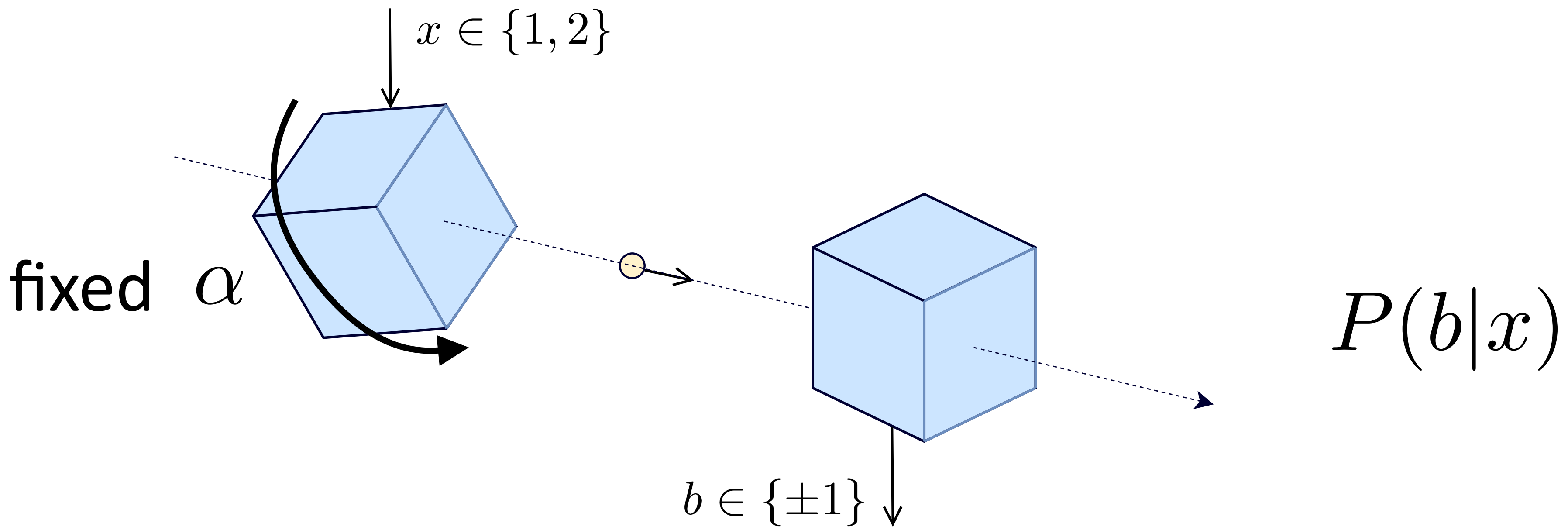

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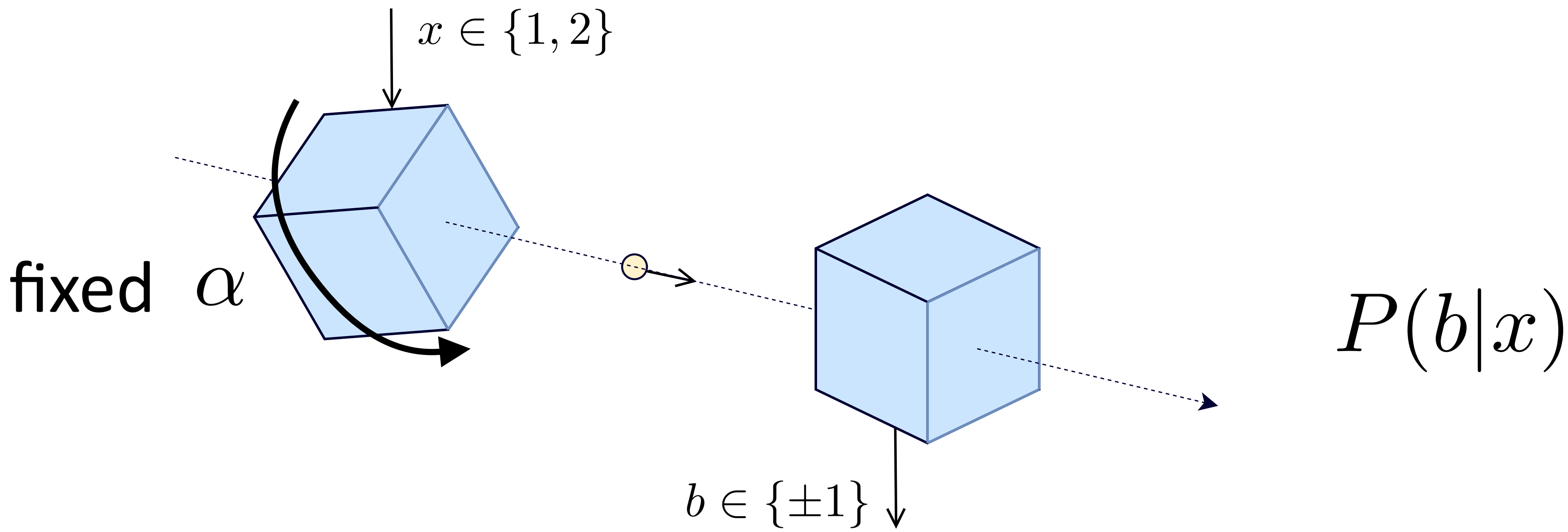

If input is x=1: do nothing to preparation device;

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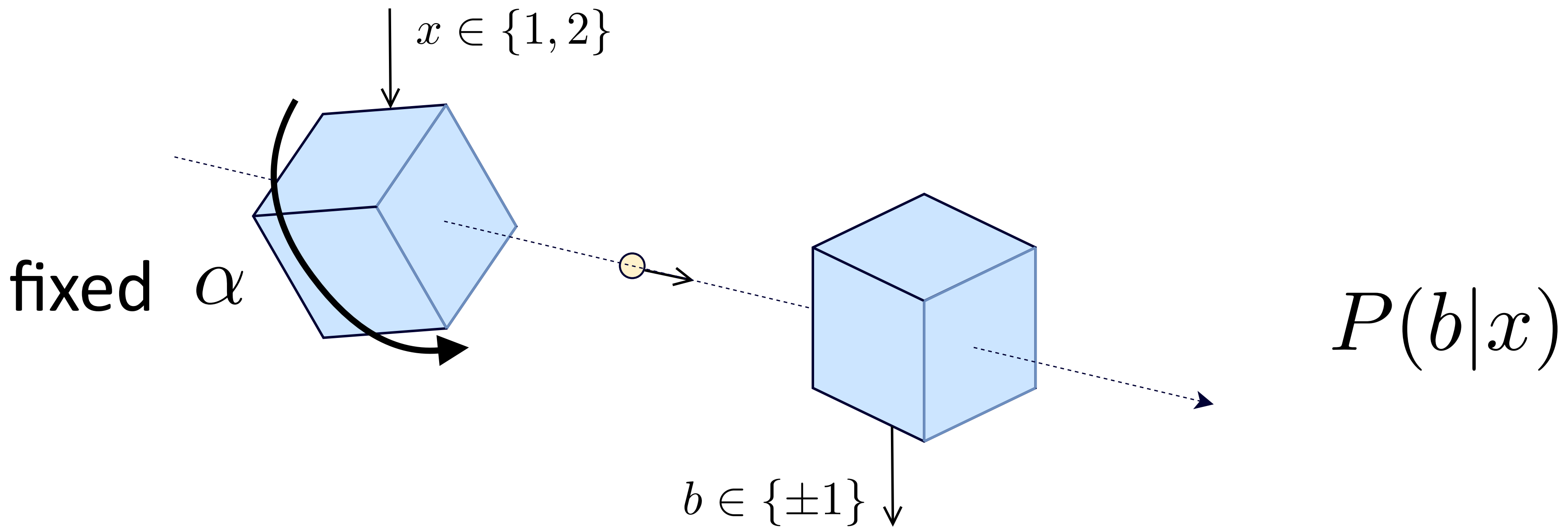

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SDI assumption: "spin" of system ≤ $\mathbf{J}$

No further assumptions on devices / system.

Do not even assume the validity of quantum theory!

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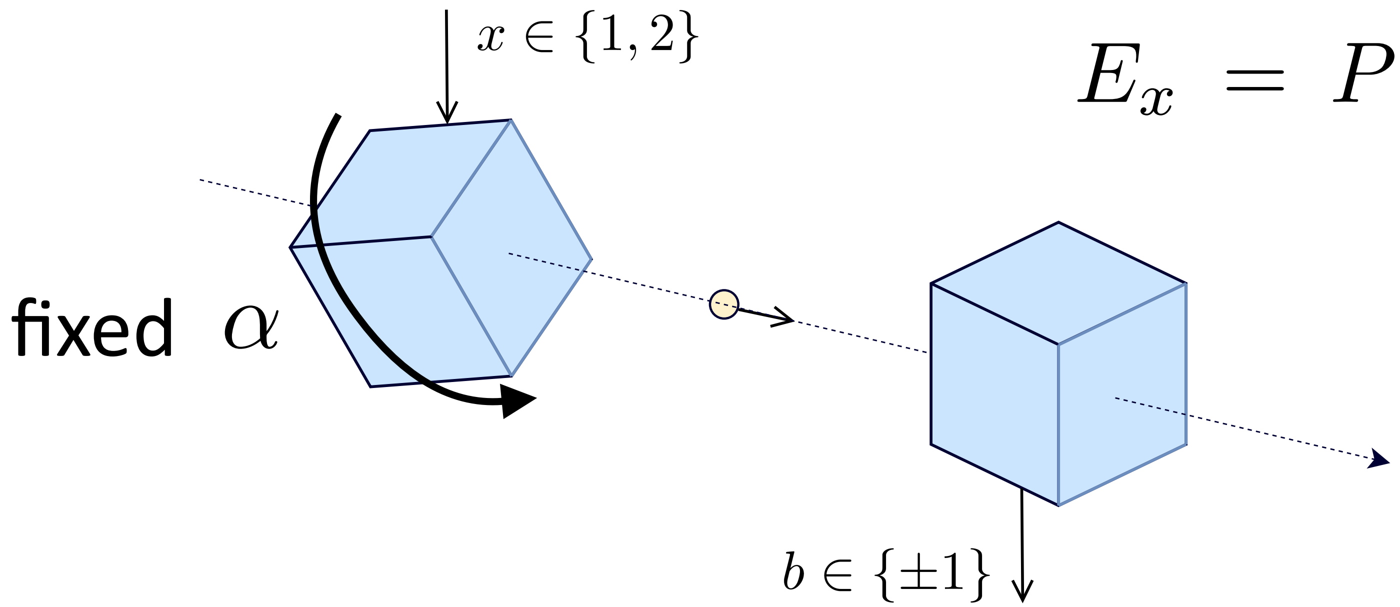

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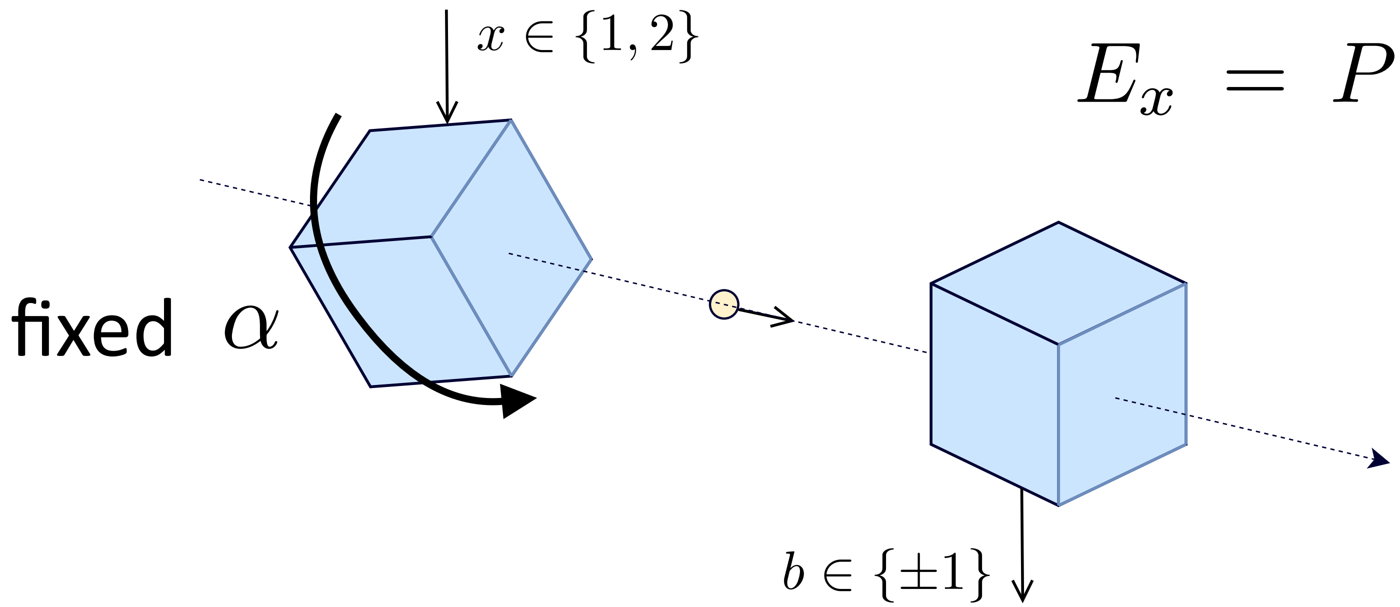

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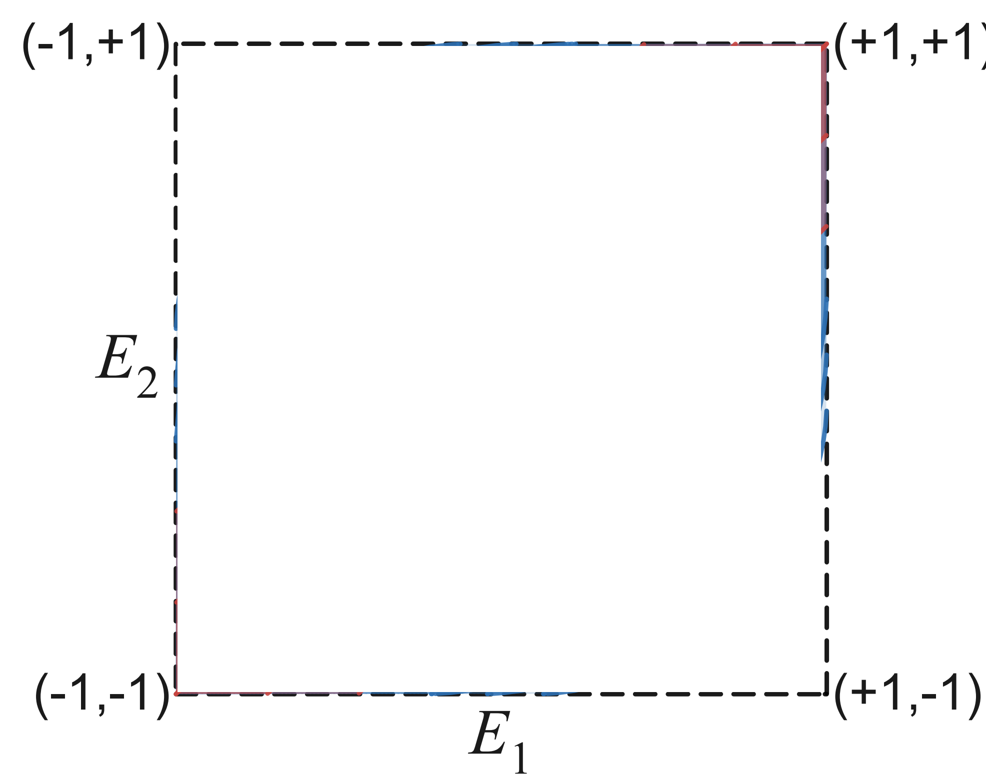

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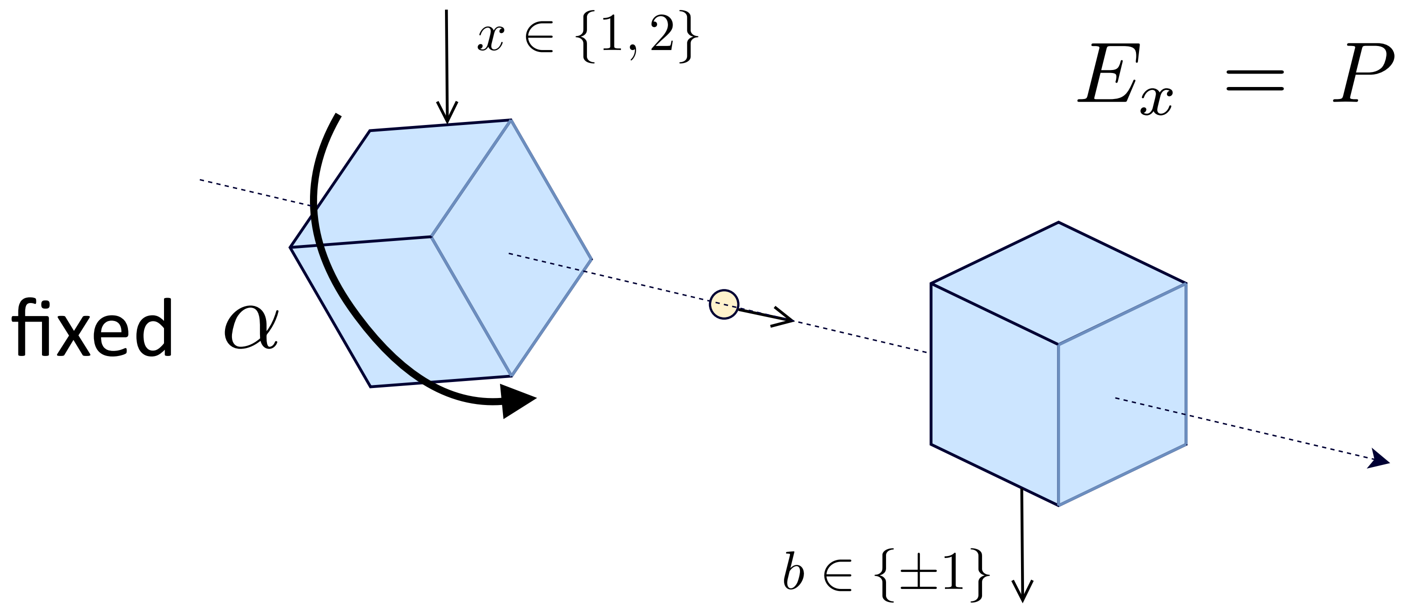

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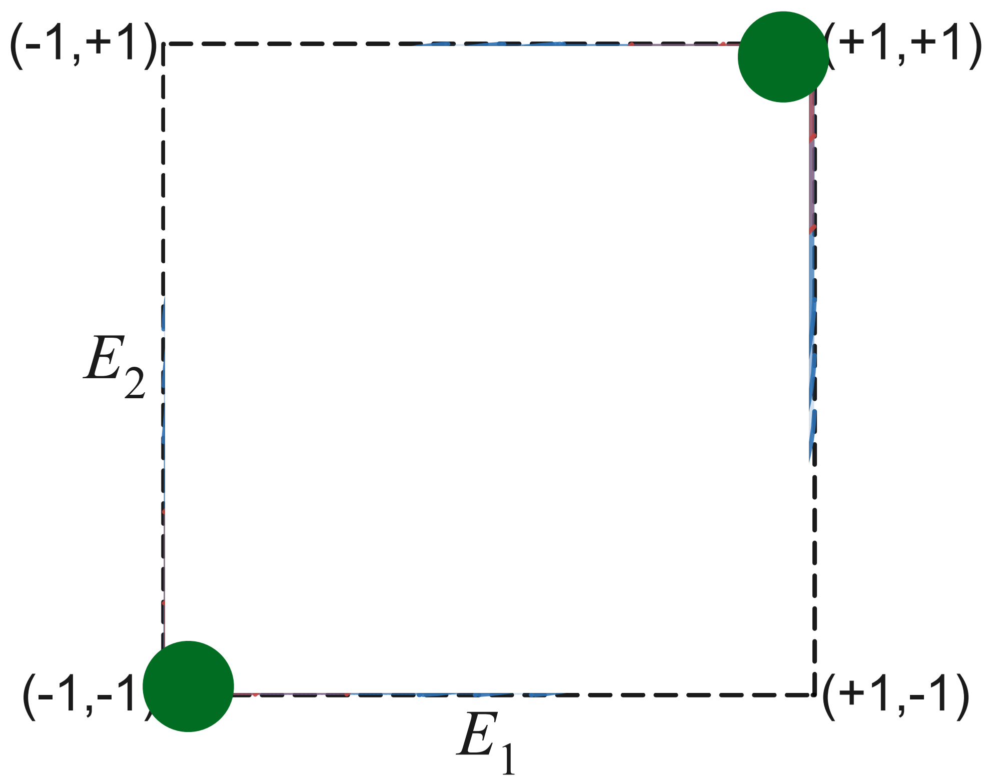

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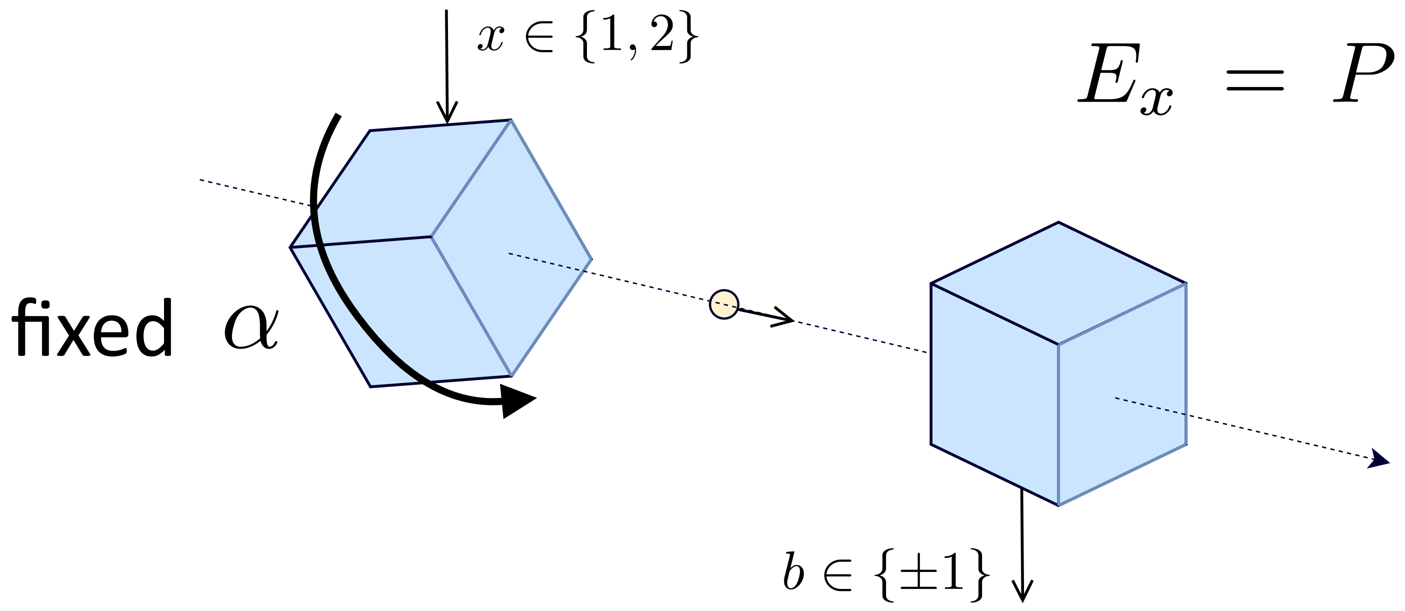

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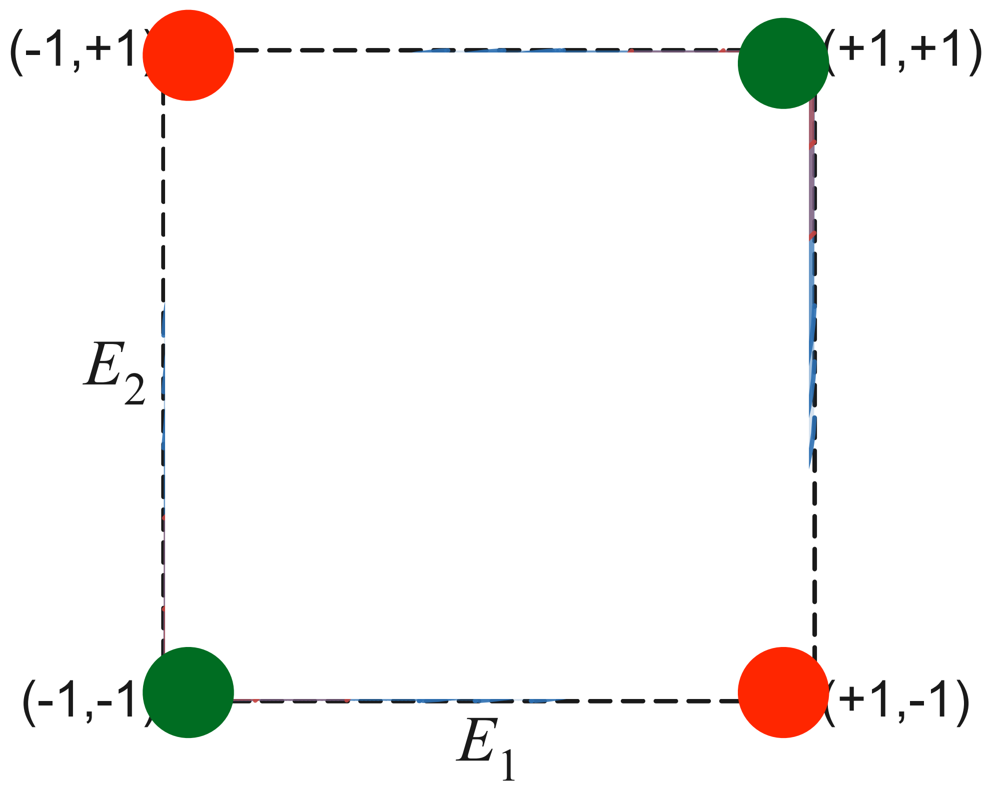

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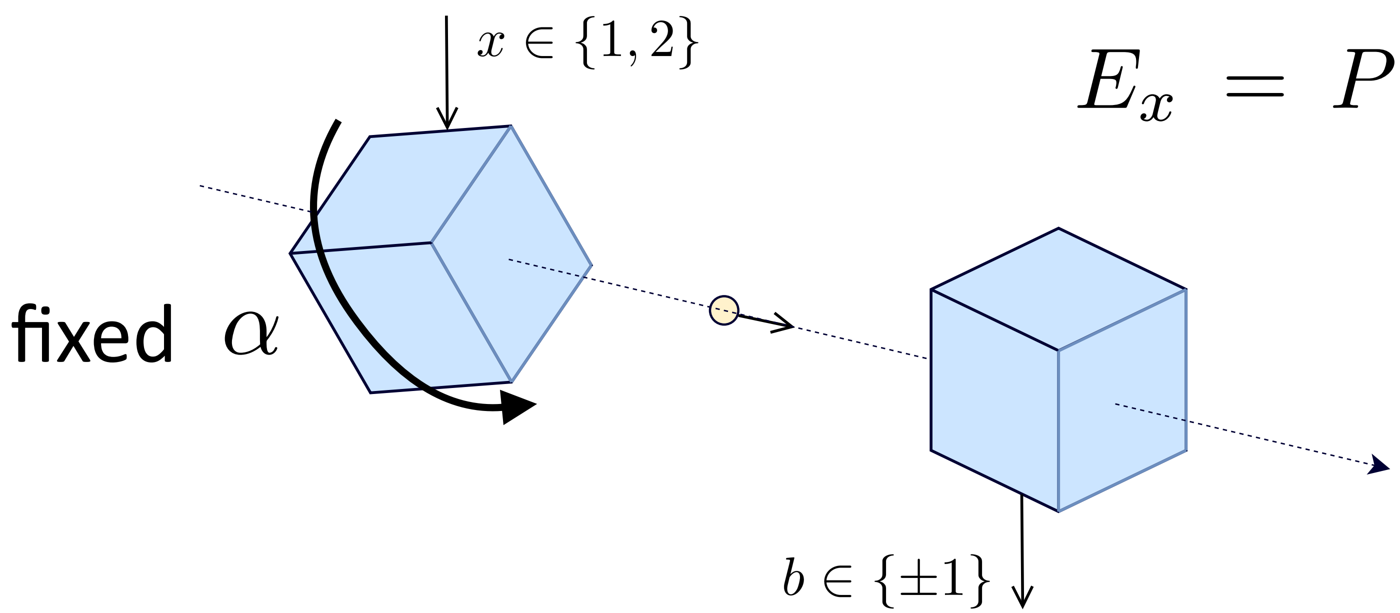

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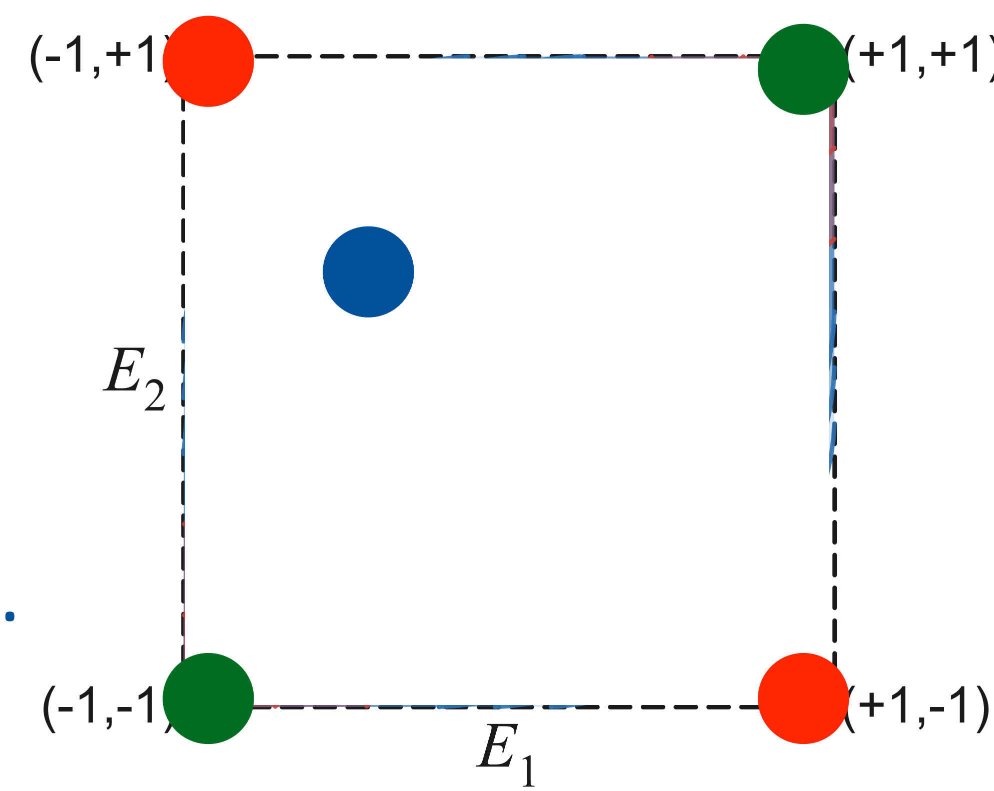

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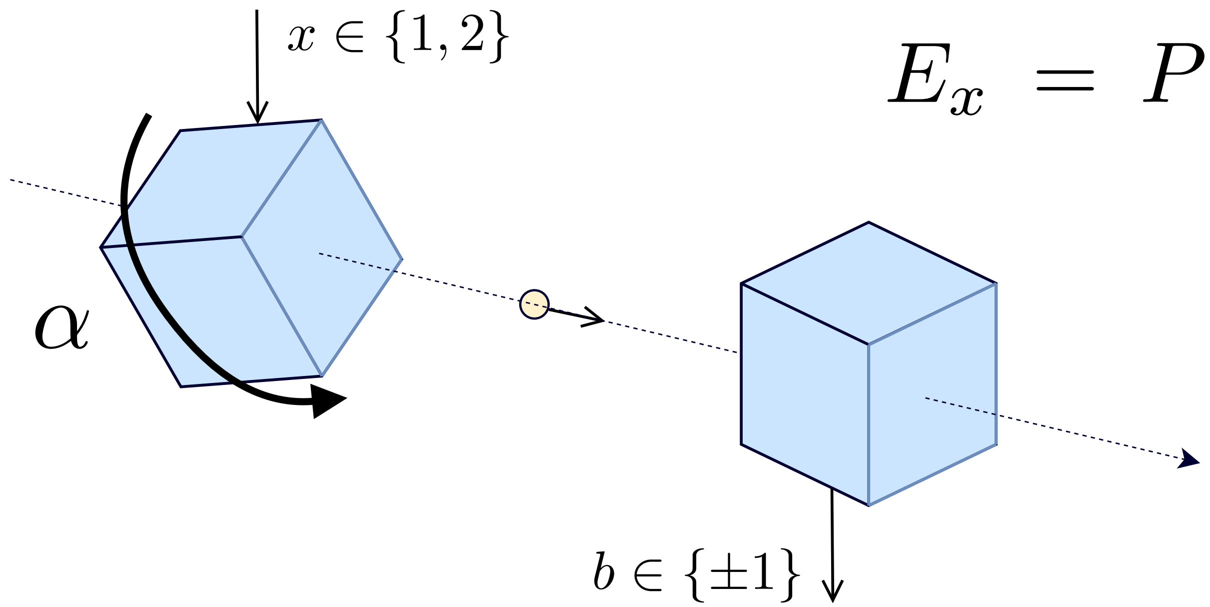

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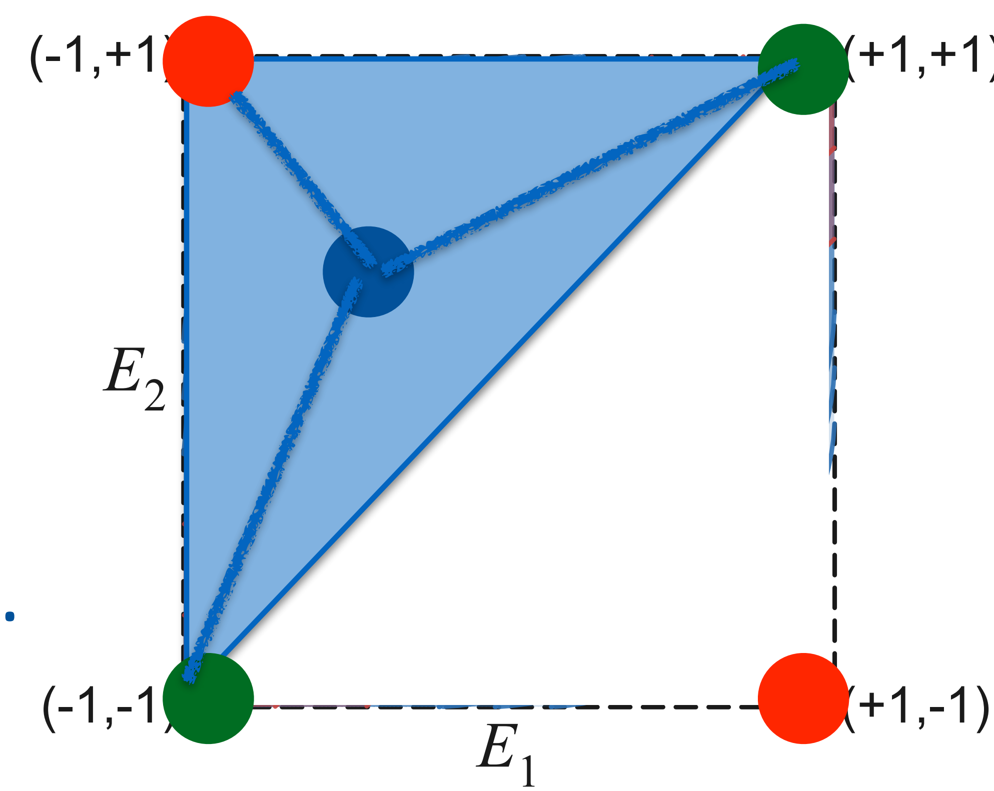

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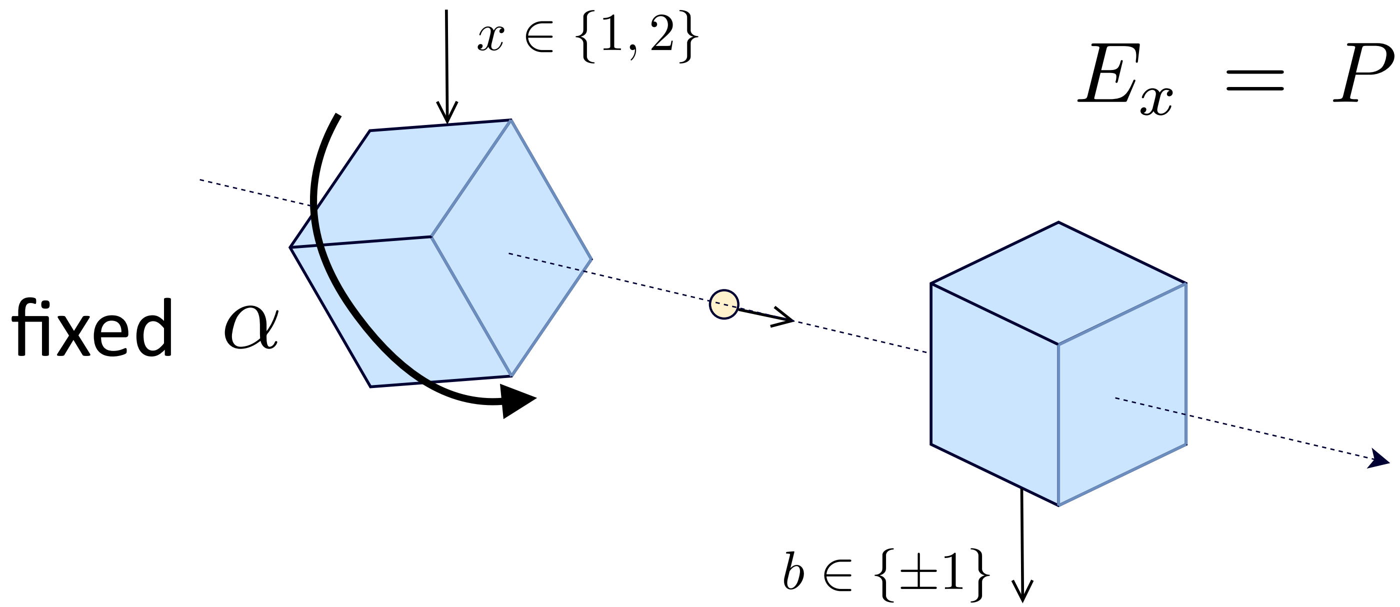

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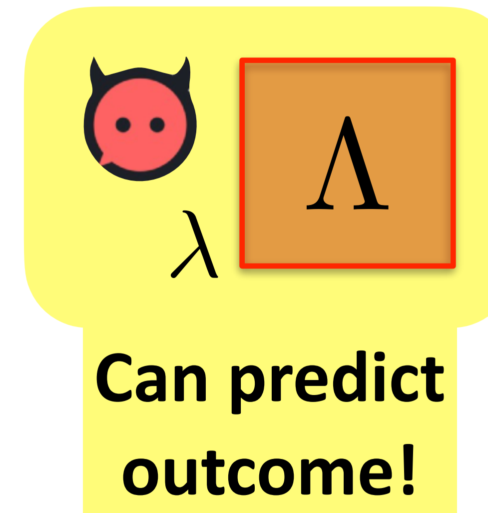

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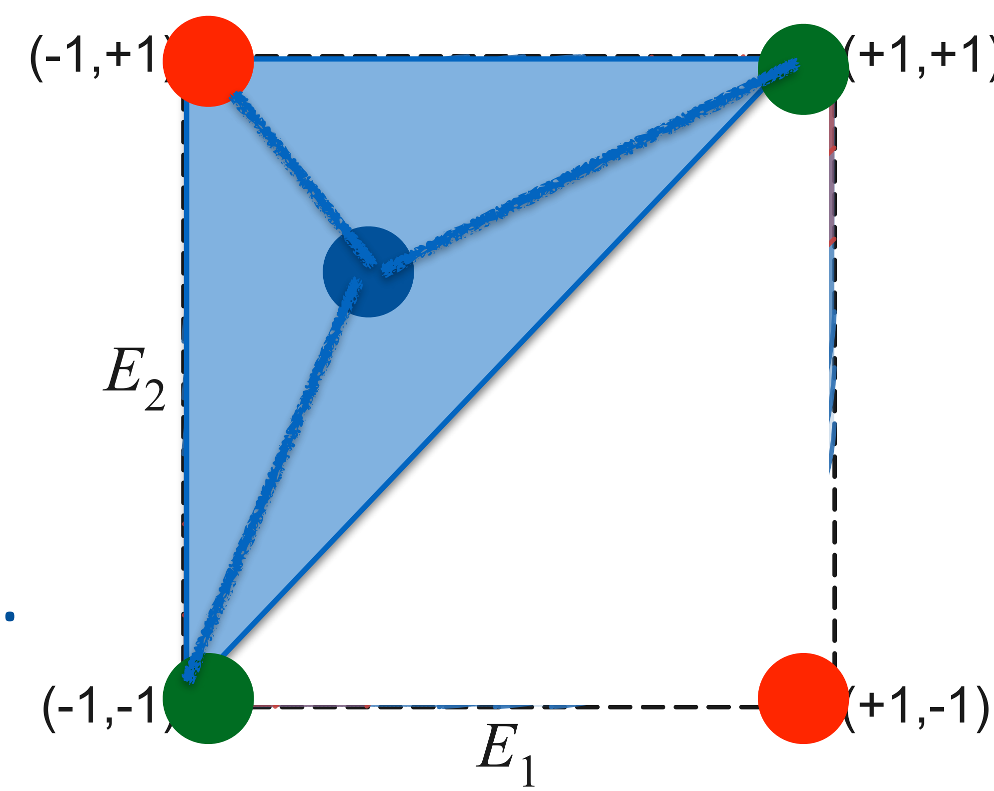

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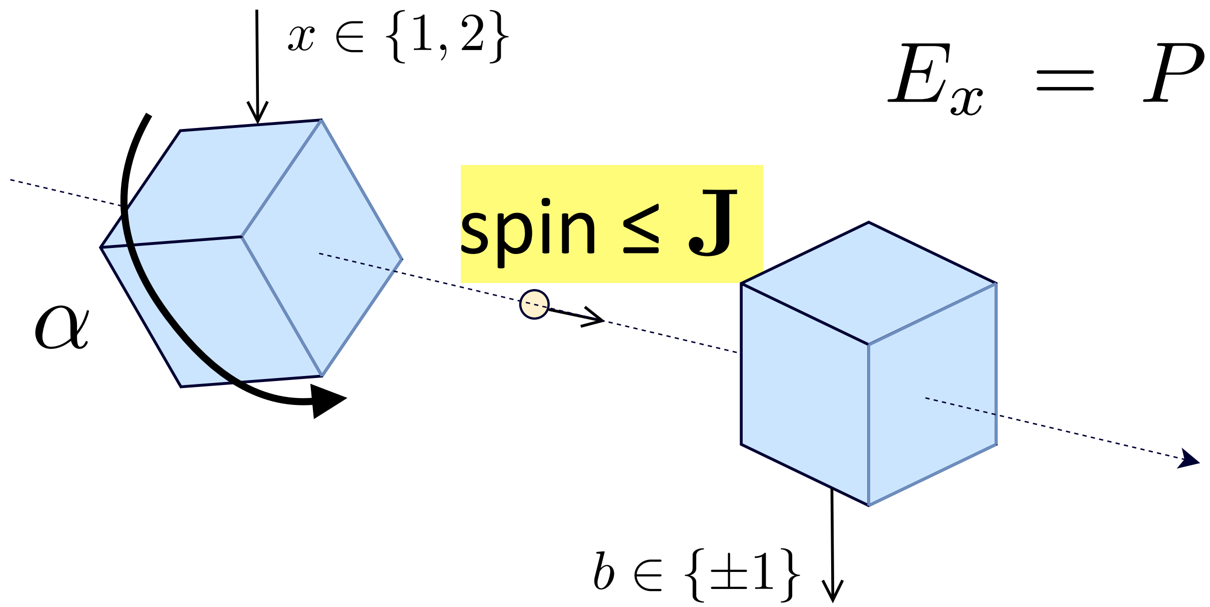

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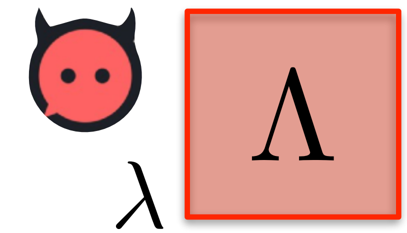
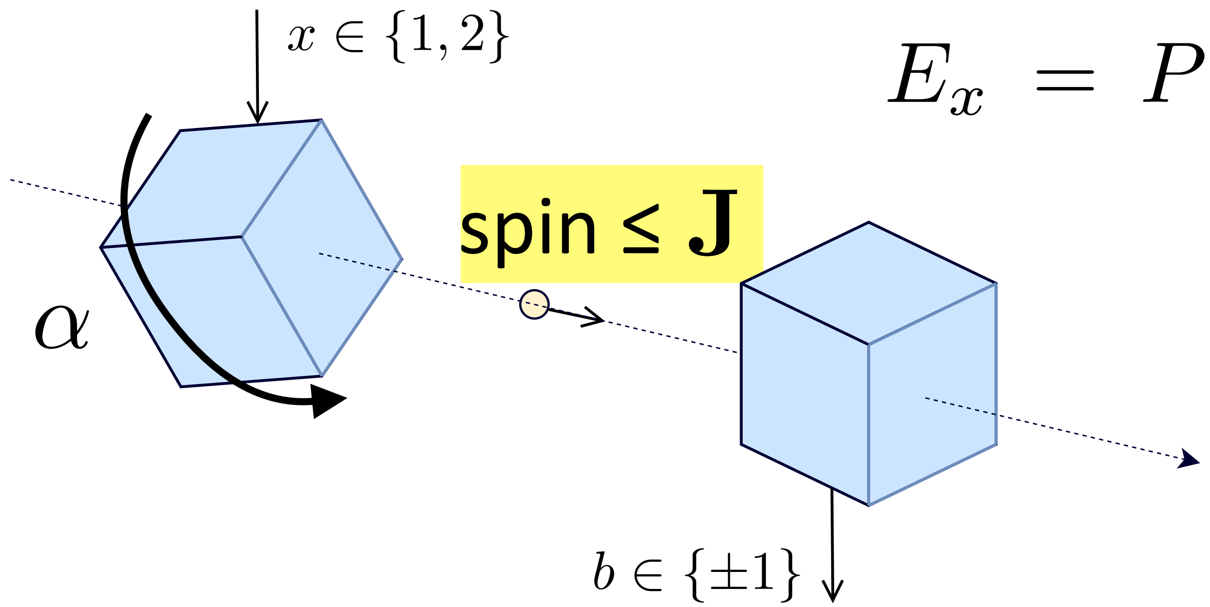


Which correlations are possible?

A theory-independent SDI randomness generator



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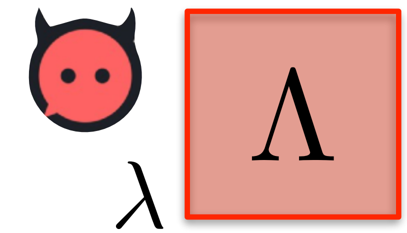

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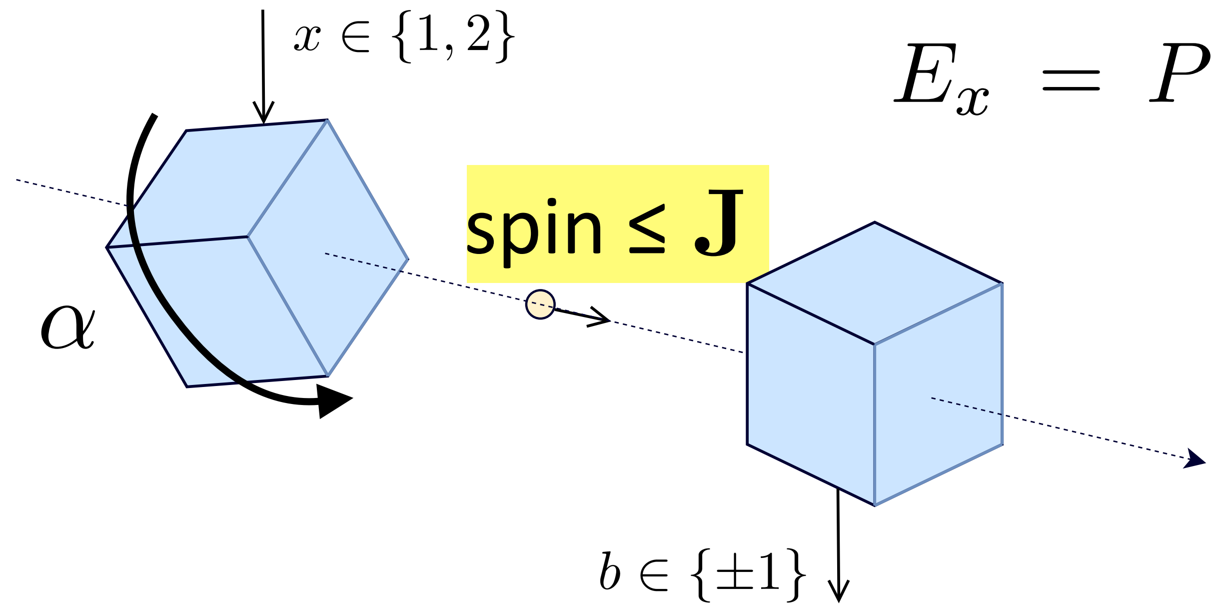
C. L. Jones, S. L. Ludescher, A. Aloy, MM, arXiv:2210.14811

using results of

T. Van Himbeeck, E. Woodhead, N. J. Cerf, R. García-Patrón, S. Pironio, Quantum **1**, 33 (2017).

A theory-independent SDI randomness generator



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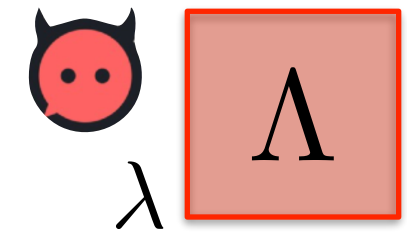

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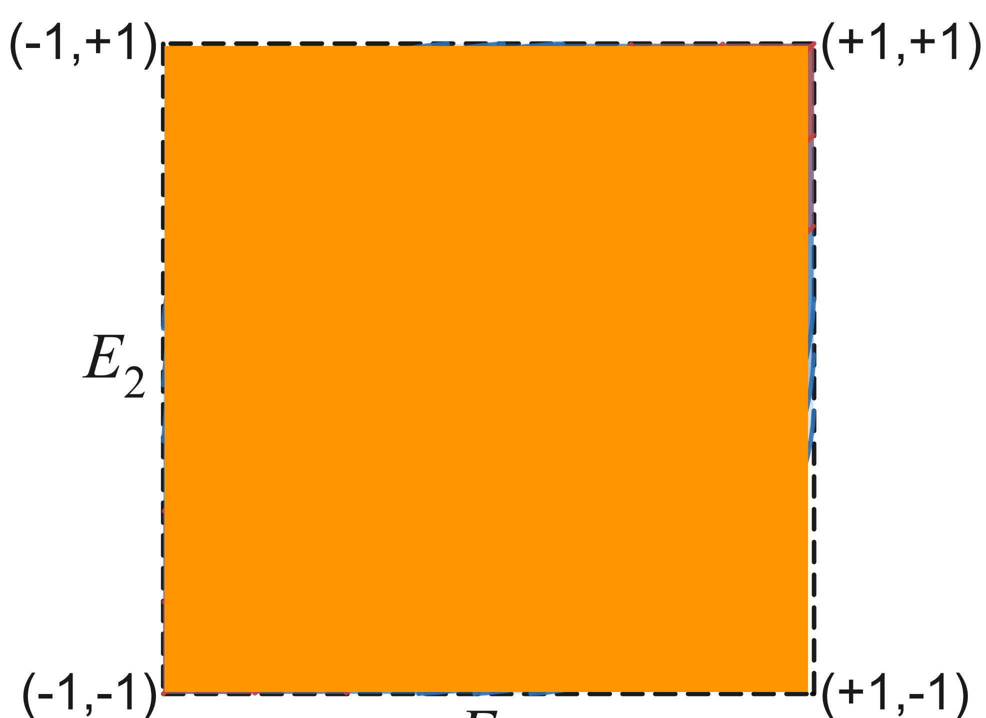
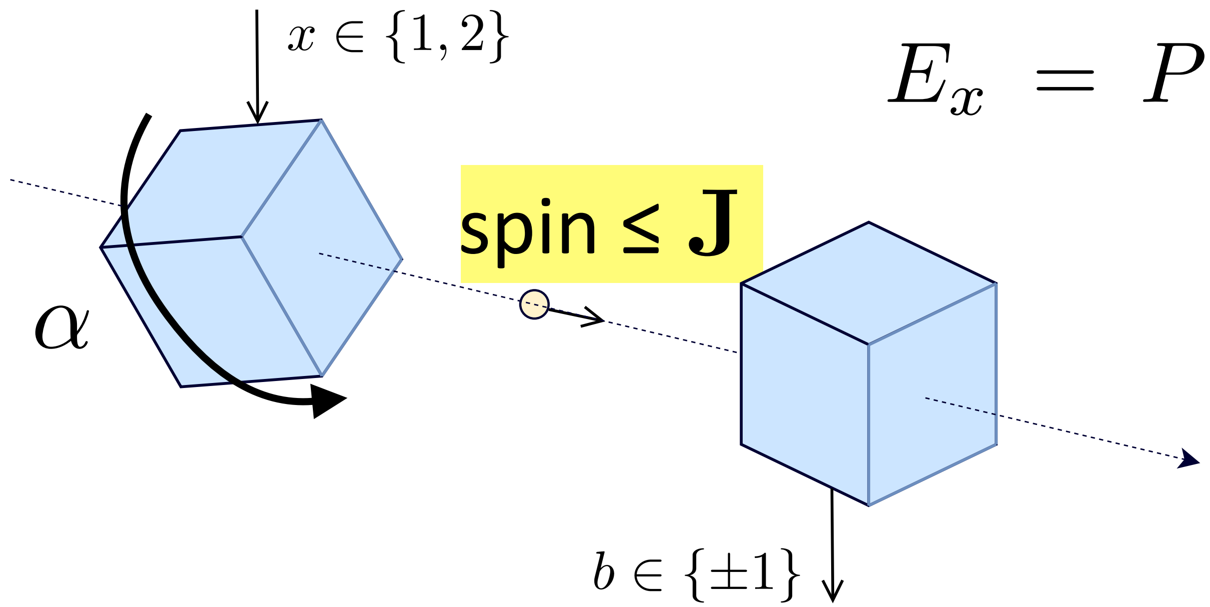


Angle $\alpha \geq \pi/(2J)$:
no certifiable randomness.

A theory-independent SDI randomness generator



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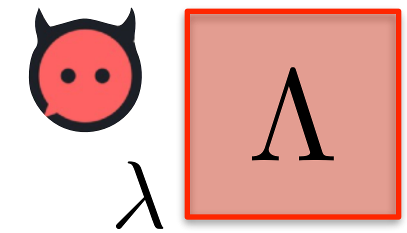

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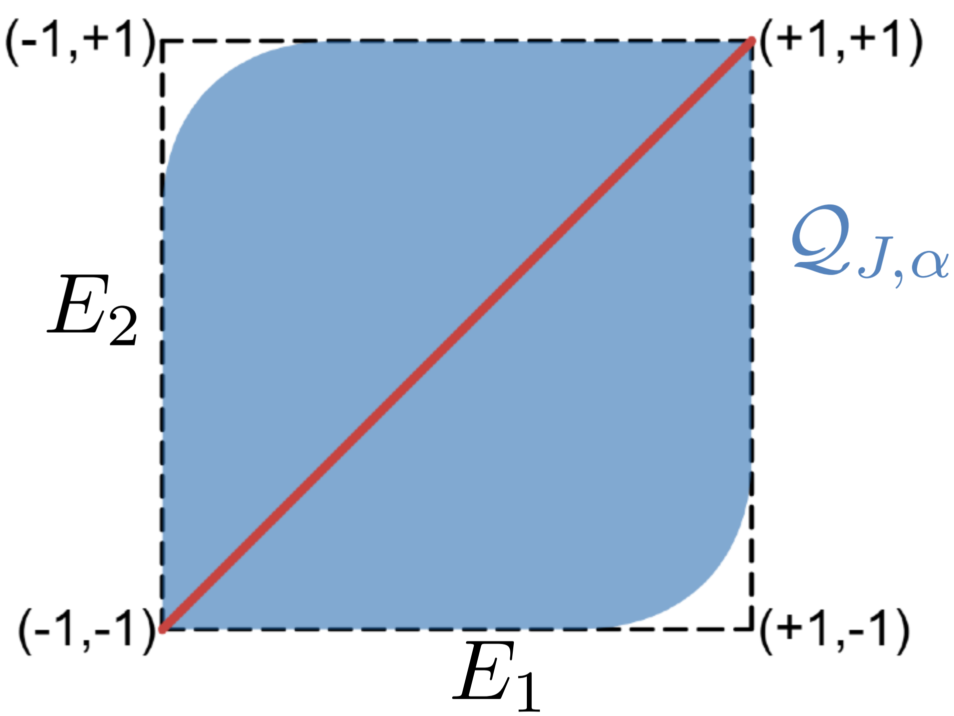

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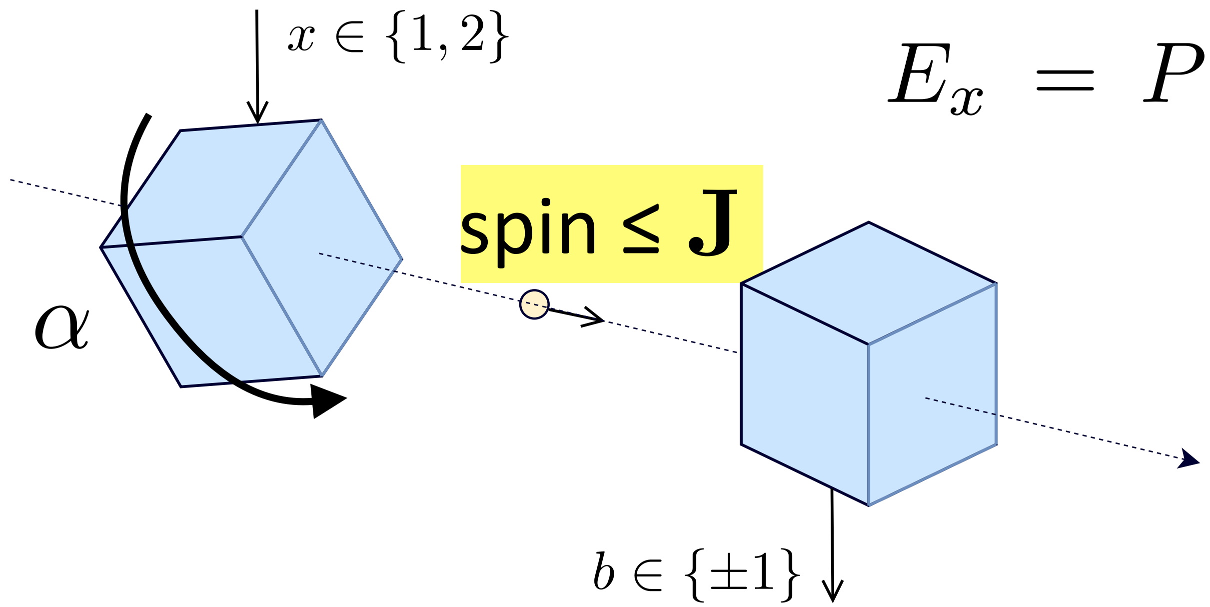
A theory-independent SDI randomness generator



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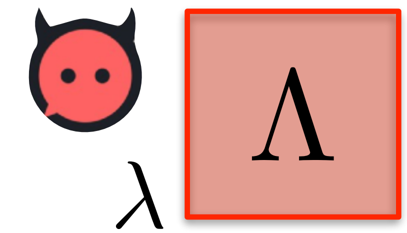

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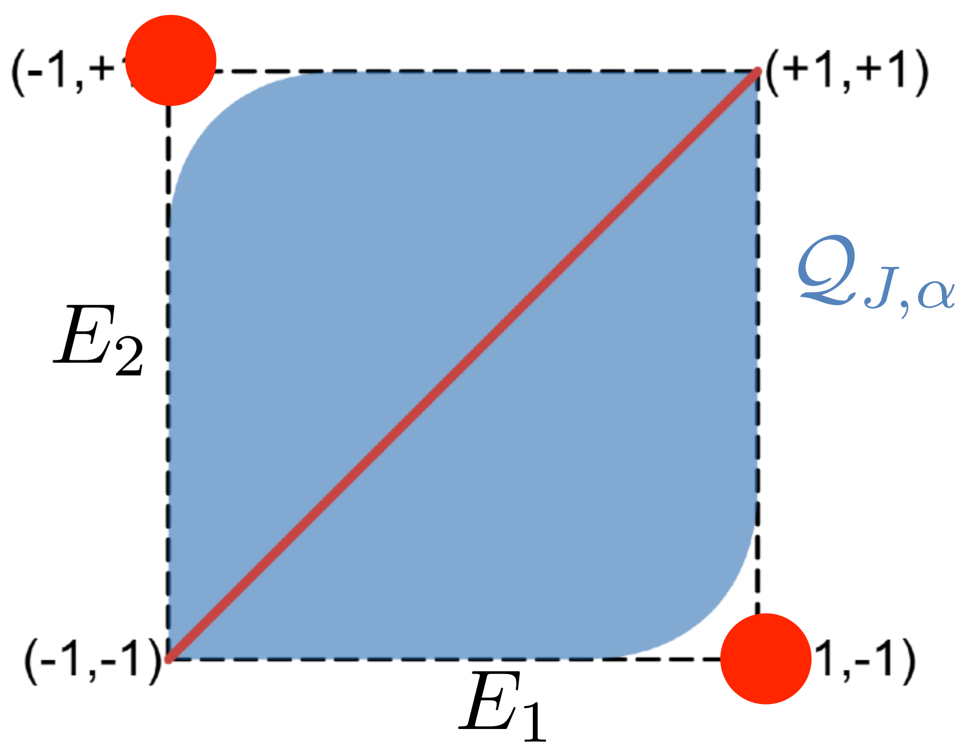

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If $|\alpha| < \pi/(2J)$, then the "interesting deterministic correlations" are forbidden...

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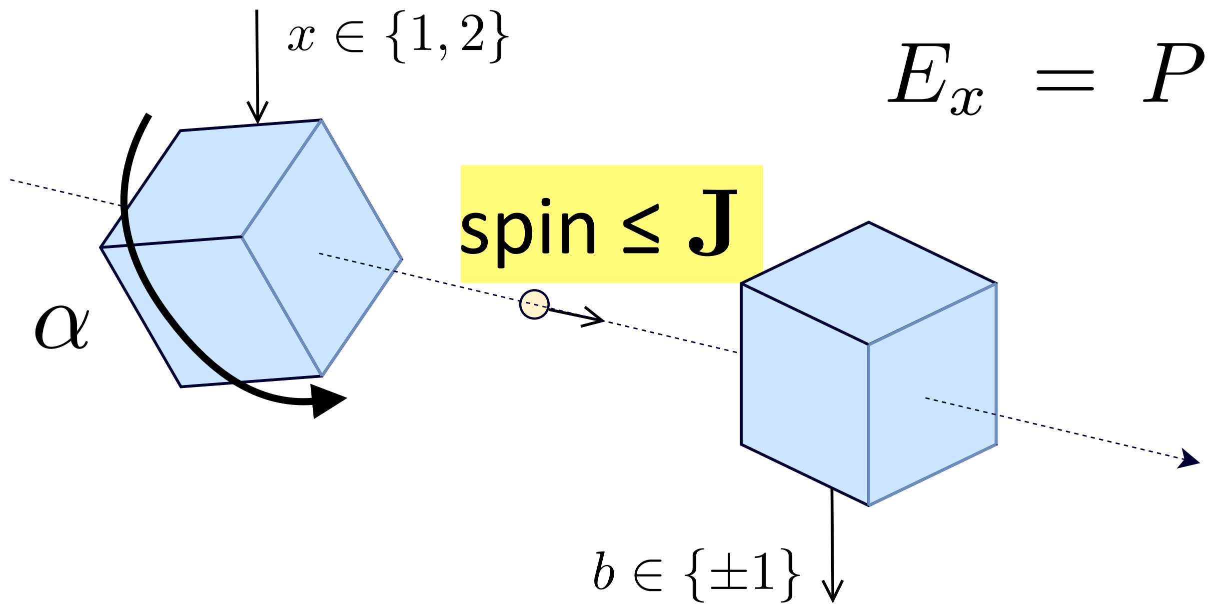

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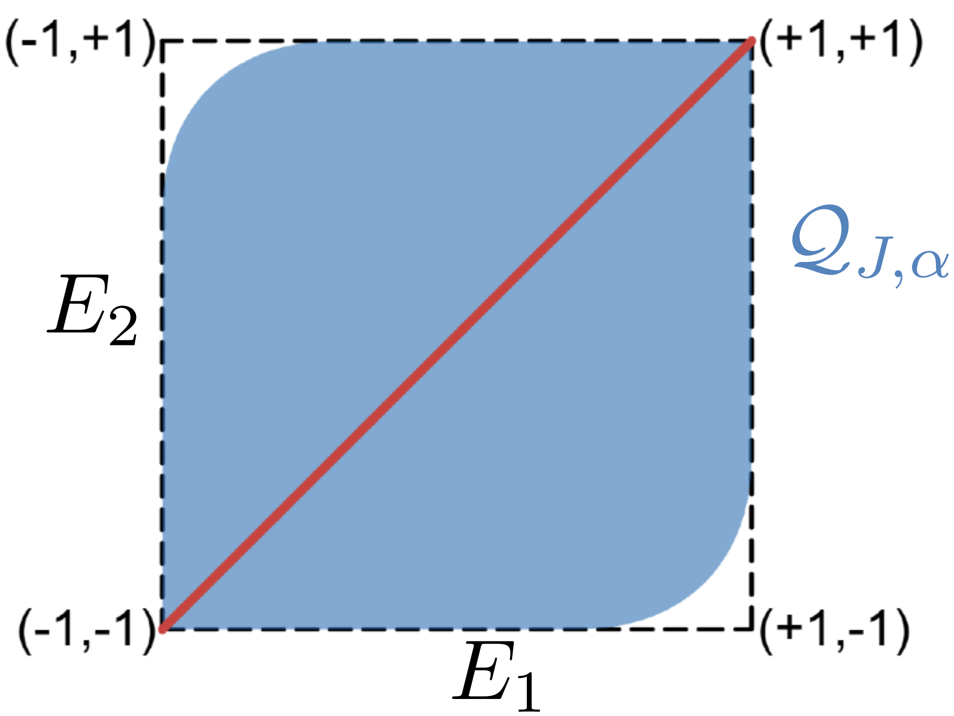

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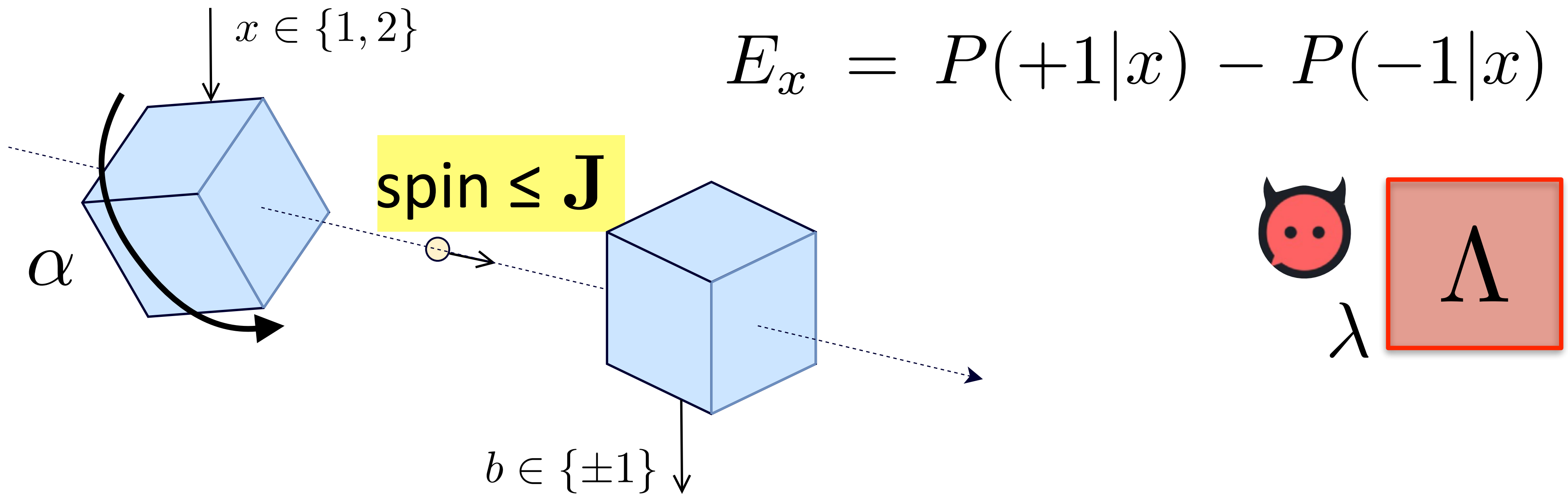
… and only correlations on the red line admit perfect outcome prediction by eavesdropper.

A theory-independent SDI randomness generator



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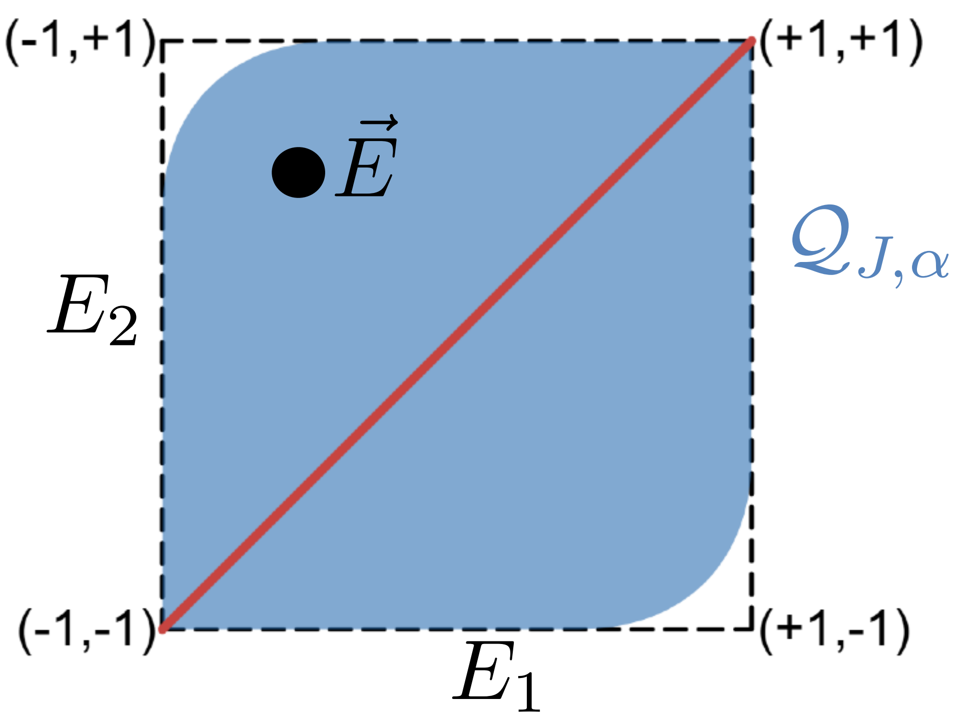

Observing some correlation $\vec{E} = (E_1, E_2)$ outside the red line thus allows us to certify randomness against the eavesdropper.

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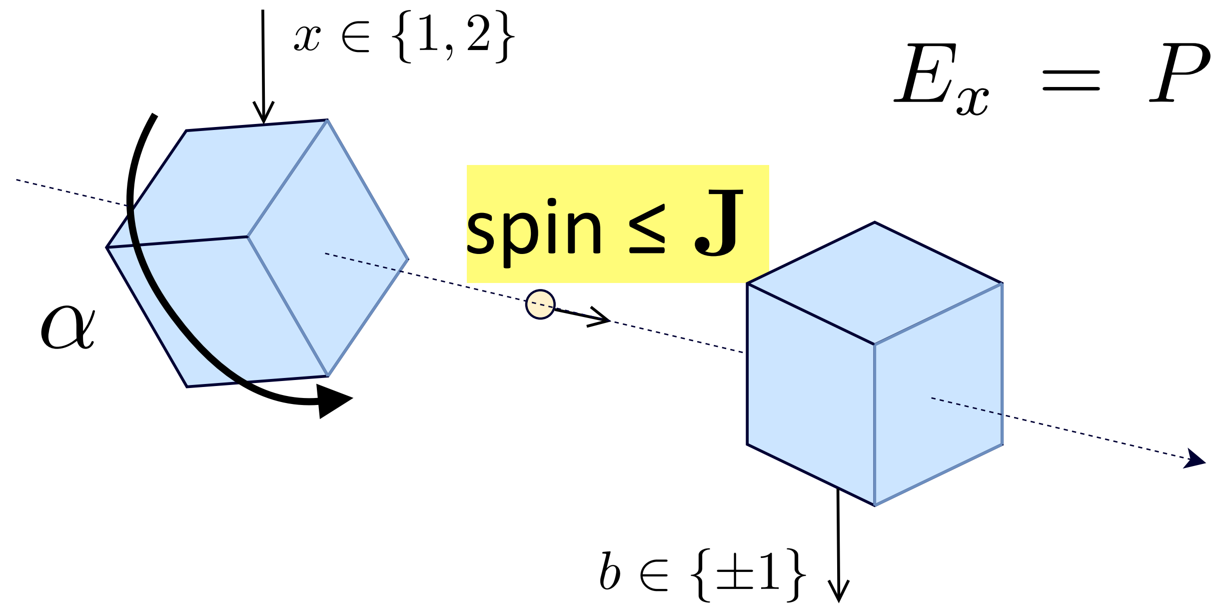

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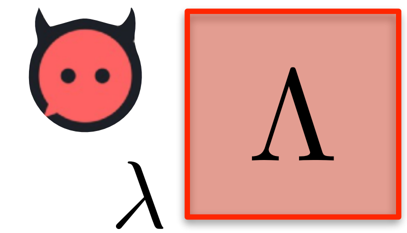

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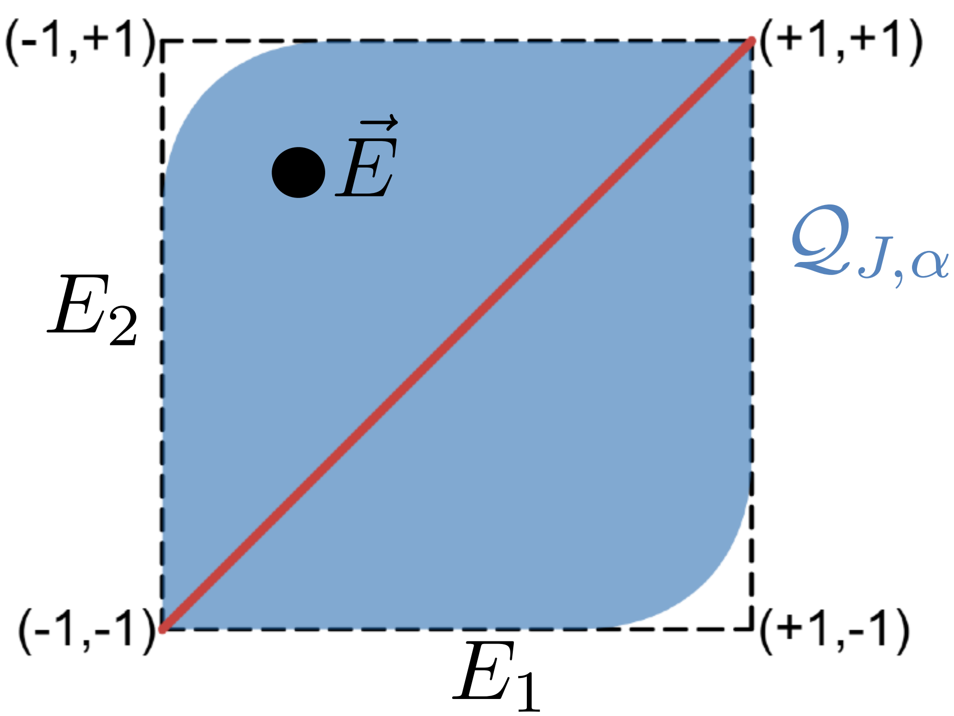

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This fact, *and* the amount of random bits, is **independent of the probabilistic theory.**

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1. Motivation (and some history)

2. “Rotation boxes” within and beyond QT

3. A metrological game and the (sub)optimality of QT

4. A theory-independent SDI randomness generator

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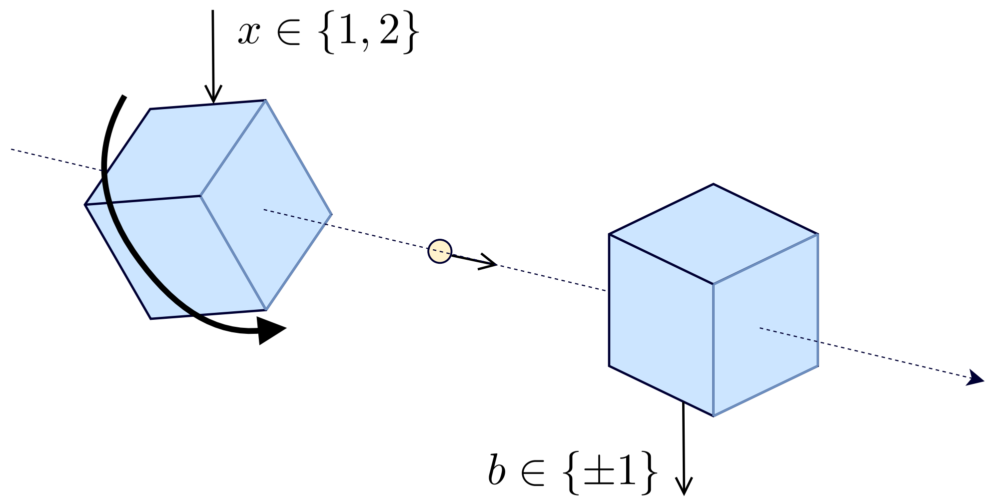
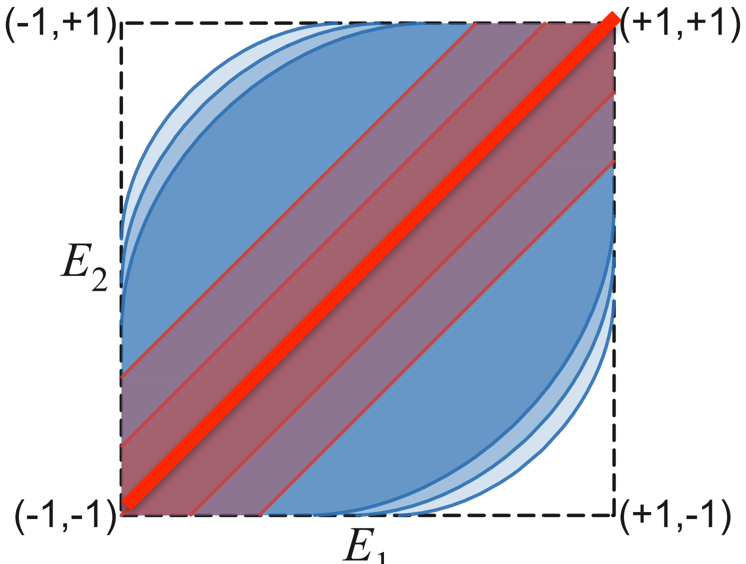
  $\mathcal{Q}_{J,\alpha} = \mathcal{R}_{J,\alpha}$. Correlations exactly determined by covariance





Thank you!

- A. Aloy, T. D. Galley, C. L. Jones, S. L. Ludescher, and M. P. Müller, *Spin-bounded correlations: rotation boxes within and beyond quantum theory*, Commun. Math. Phys. **405**, 292 (2024).
- C. L. Jones, S. L. Ludescher, A. Aloy, and M. P. Müller, *Theory-independent randomness generation from spatial symmetries*, arXiv:2210.14811.