Single shot q-thermo, VL 2

Course homepage: www.mpmueller.net/lecture2.html

No lecture/tutorial/exercise sheet in the week of Nov. 3-7!

Tutorial: Tuesday (Di.) 14:15-15:45,
Philosophenweg 12, room 060

Credit points for 60% of homework.

Goal: $\ll$ work than theor. Phys. [redacted] exercises

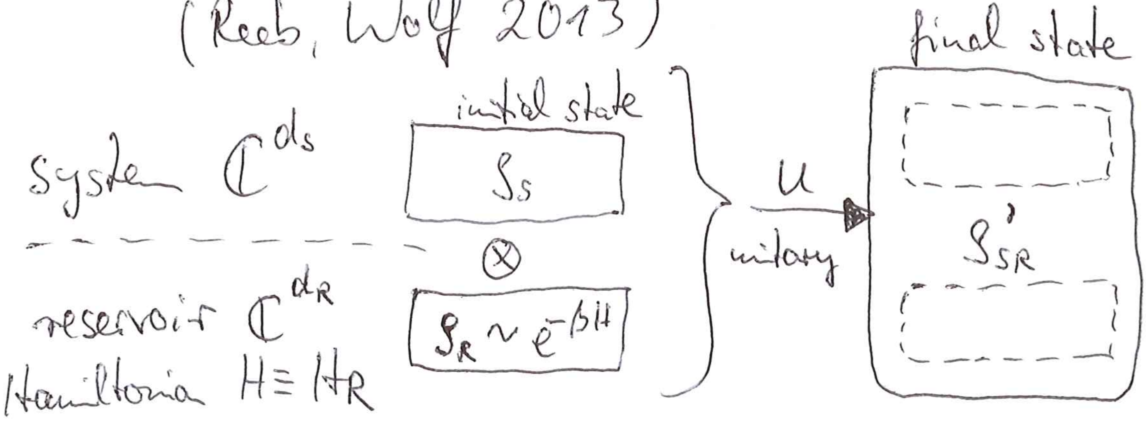
groups of 2 or 3 people.

any further organisational questions?

1. Landauer's Principle: rigorous derivations

1.1. A quantum-information-theoretic version

(Reeb, Wolf 2013)



$\rho'_{SR} = U(\rho_S \otimes \rho_R)U^\dagger$

$\rho'_S = \mathrm{Tr}_R\, \rho'_{SR}$

$\rho'_R = \mathrm{Tr}_S\, \rho'_{SR}$

$\Delta S_S := S(\rho_S) - S(\rho'_S)$ entropy reduction in system

$\Delta S_R := S(\rho_R) - S(\rho'_R)$ " " " reservoir

(1) $\Delta Q := \mathrm{tr}[H\rho'_R] - \mathrm{tr}[H\rho_R]$ heat increase in reservoir

Theorem: $\boxed{\Delta Q \geq k_B T \, \Delta S_s}$ $\beta = 1/(k_B T)$

In more detail

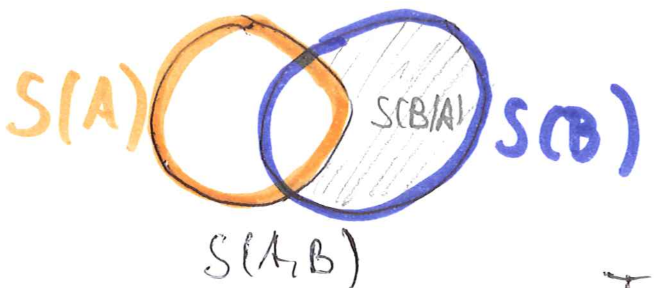
$$\Delta Q = k_B T \Big(\Delta S_s + \underbrace{I(S' : R')}_{\substack{\text{mutual information} \\ \geq 0}} + \underbrace{D(\rho_R' \| \rho_R)}_{\substack{\text{relative entropy} \\ \geq 0}} \Big)$$

Remarks:

- This is work extraction on average (exp. value of $H = H_R$; to be determined in many runs of the experiment). "Single-shot" approach will be: run only once, and aim at prob $\geq 1-\varepsilon$ for dissipation of energy $\leq E$).
- $I(S' : R')$ is a measure of correlations in the final state;

$$\begin{aligned} I(A:B) &= S(A) + S(B) - S(AB) \\ &\equiv S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \\ &= S(B) - S(B|A) \qquad \geq 0 \end{aligned}$$

where $S(B|A) := S(A,B) - S(A)$ is the conditional entropy: remaining uncertainty about B if we learn about A



$\Rightarrow I(A:B)$ = how much learning A helps us to learn about B.

$I(A:B) = I(B:A)$

Theorem says: correlations are costly!

Ex.: $\rho_{AB} = \rho_A \otimes \rho_B$ (uncorrelated) $\Rightarrow S(AB) = S(A) + S(B)$

$\Rightarrow I(A:B) = 0.$

$\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ (classical correlation)

$\Rightarrow \rho_A = \rho_B = \frac{1}{2}\mathbb{1} \Rightarrow S(AB) = S(A) = S(B) = \log 2$

$\Rightarrow I(A:B) = \log 2 \; [=1 \text{ if } \log = \log_2]$

"one bit of mutual information".

Proof: First we show the "second law lemma"

$$\underbrace{(-\Delta S_S)}_{\text{entropy increase in system}} + \underbrace{(-\Delta S_R)}_{\text{entropy increase in reservoir}} = I(S':R') \geq 0 .$$

$$-\Delta S_S - \Delta S_R = S(\rho_S') - S(\rho_S) + S(\rho_R') - S(\rho_R)$$

$$= S(\rho_S') + S(\rho_R') - S(\rho_S \otimes \rho_R)$$

$$= S(\rho_S') + S(\rho_R') - S(\rho_{SR}')$$

$S(U\rho U^\dagger) = S(\rho)$

$$= I(S':R') \geq 0 .$$

$$\Rightarrow \Delta S_S + I(S':R') = -\Delta S_R = S(\rho_R') - S(\rho_R)$$

$$\boxed{\begin{array}{l} \rho_R = e^{-\beta H}/Z \text{ is thermal state} \\ \Rightarrow S(\rho_R) = \beta\, \mathrm{tr}(\rho_R H) + \ln Z^{*} \end{array}}$$

$$= -\mathrm{tr}(\rho_R' \ln \rho_R') - \underline{\beta\, \mathrm{tr}(\rho_R H)} - \ln Z$$
$$+ \underline{\beta\, \mathrm{tr}(\rho_R' H)} - \beta\, \mathrm{tr}(\rho_R' H)$$
$$= \underline{\beta \Delta Q} - \mathrm{tr}\left(\rho_R' \ln \rho_R' - \rho_R' \ln \frac{e^{-\beta H}}{Z}\right)$$
$$= \beta \Delta Q - D(\rho_R' \| \rho_R). \qquad \square$$

2. The resource theory of nonuniformity

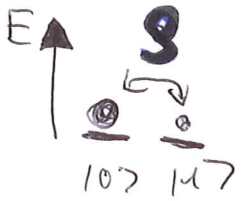
2.1. Motivating examples

Standard Landauer erasure:

$$\rho = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|$$
$$\downarrow$$
$$\rho' = |0\rangle\langle 0|$$

Work cost of bit reset: $\langle W\rangle = k_B T \ln 2$.

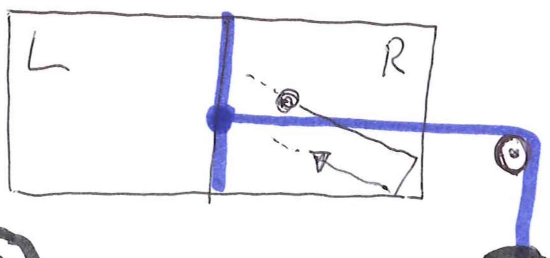
What if ρ is <u>any</u> state?



We will show later on: if we have many particles in state ρ, the average work cost <u>per particle</u> is $\langle W\rangle = k_B T\, S(\rho)$

von Neuman entropy

Complementary picture: the Szilard engine

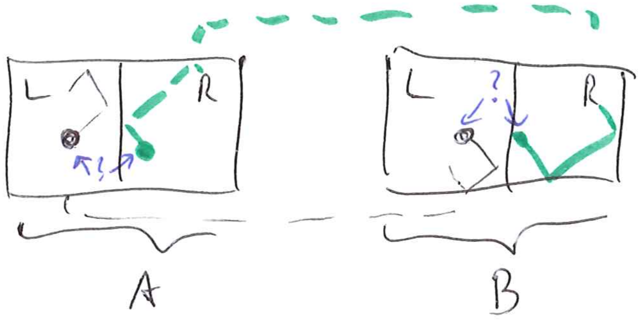


$$\rho = |R\rangle\langle R|$$
$$\downarrow$$
$$\rho' = \tfrac{1}{2}|L\rangle\langle L| + \tfrac{1}{2}|R\rangle\langle R|$$

work value

 Many particles $\Longrightarrow$ average work ~~value~~ <u>per particles</u>: $\langle W\rangle = k_B T(1 - S(\rho))$

Consider two correlated Szilard engines (cf. Bennett 2003)



$$\rho_{AB} = \frac{1}{2}|LL\rangle\langle LL| + \frac{1}{2}|RR\rangle\langle RR|$$

the local states are $\rho_A = \mathrm{Tr}_B\, \rho_{AB} = \frac{1}{2}|L\rangle\langle L| + \frac{1}{2}|R\rangle\langle R| = \frac{1}{2}\mathbb{1}$,

$$\rho_B = \frac{1}{2}\mathbb{1}.$$

$\Rightarrow$ By **acting** locally, we can extract zero work.

<u>But</u>: we can reversibly implement the CNOT operation

$$\begin{aligned} LL &\to LL \\ LR &\to LR \\ RL &\to RR \\ RR &\to RL \end{aligned}$$

- unitary $U|RL\rangle = |RR\rangle$ etc.

$$\hat{\rho}_{AB} := U \rho_{AB} U^\dagger = \frac{1}{2}|LL\rangle\langle LL| + \frac{1}{2}|RL\rangle\langle RL| = \left(\frac{1}{2}\mathbb{1}\right) \otimes |L\rangle\langle L|$$

Now we can extract $kT \log 2$ of work from engine B!

$\Rightarrow$ Strategy:

$$\boxed{\begin{array}{l}\text{1. Extract as many pure (qu)bits as possible (via reversible transformations).} \\ \text{2. Use them for work extraction.}\end{array}}$$

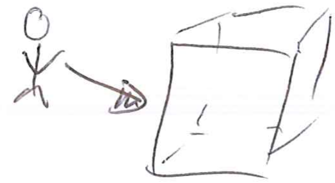
Remarks for step 1:

- $|L\rangle$ and $|R\rangle$ assumed to have the same energy $\Rightarrow$ Hamiltonian fully degenerate.
- Laws of nature restrict us to <u>reversible</u> transformations (and this is the <u>only</u> hard restriction).

• We can always introduce extra engines in "thermal" state $\rho = \frac{1}{2}|L\rangle\langle L| + \frac{1}{2}|R\rangle\langle R| = \frac{1}{2}\mathbb{1}$ "for free"

2.2 Definition of the resource theory of nonuniformity

<u>Resource theory</u>: an agent has limited control over some [redacted] physical system



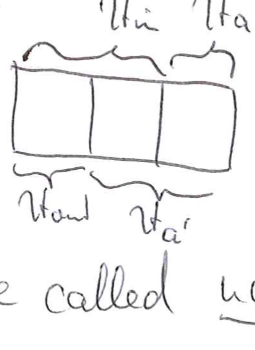
R.T. of nonuniformity: The agent is given some quantum state ρ_{in} on a finite-dim. Hilbert space $\mathcal{H}_{in}$.
Then he may

- introduce an extra system $\mathcal{H}_a$ in the maximally mixed state $\gamma_a := \frac{\mathbb{1}}{d_a}$, $d_a = \dim \mathcal{H}_a$, "for free",
- apply arbitrary unitary maps on arbitrary subsystems, and
- disregard subsystems, i.e. trace/marginalize over subsystems.

Then every map he can implement by composing [redacted] those operations can be written in the form

$$\rho_{out} = \boxed{\mathcal{E}(\rho_{in}) = \mathrm{Tr}_{a'}\left[U(\rho_{in} \otimes \gamma_a)U^\dagger\right]} \quad (*)$$

where $\mathcal{H}_{in} \otimes \mathcal{H}_a = \mathcal{H}_{out} \otimes \mathcal{H}_{a'}$ and U is an arbitrary unitary
(For example with 3 systems:



)

 <u>Def.</u>: Maps of the form (*) are called <u>noisy operations</u>.

Remarks:

- Max. mixed states γ are free, but "non-uniform" states (e.g. pure states) are valuable resources.
  $\rightarrow \mathcal{E}(\mathbb{1}/d_{in}) = \mathbb{1}/d_{out}$ ($\mathcal{E}$ is a unital quantum channel).
- Assumes perfect control (implementation of arbitrary U)
  $\rightarrow$ will explore ultimate limits of what is possible in principle.
- Main question: given ρ and σ, when does there exist a noisy operation $\mathcal{E}$ with $\mathcal{E}(\rho) = \sigma$?
  i.e. $\rho \xrightarrow{\text{noisy}} \sigma$
- Extraction of pure bits $|0\rangle \in \mathbb{C}^2$ from ρ:
  ask for largest integer n such that $\rho \xrightarrow{\text{noisy}} |0\rangle\langle 0|^{\otimes n}$
  possibly up to some error [redacted] $\varepsilon > 0$.

[redacted]

In the following, we will answer this!

- Later on: introduce Hamiltonians H to systems, and demand energy conservation
  $\rightarrow$ "resource theory of athermality",
  can explicitly compute extractable work without assuming Szilard engines / Landauer's Principle.

1st idea: arbitrary unitaries allow us to <u>diagonalize</u> everything $\rightarrow$ compute classically!