# Single-shot thermo, lecture 3

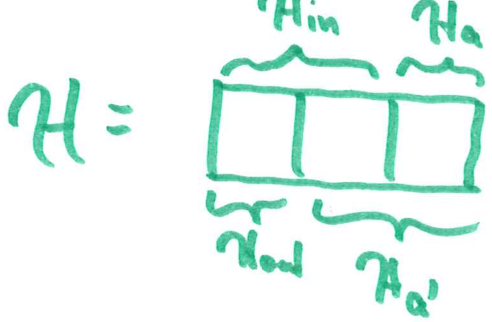
## 2.2. Definition of the resource theory of nonuniformity

If $\mathcal{H}$ is a Hilbert space, $B(\mathcal{H})$ denotes the bounded operators on it.
For us, all Hilbert spaces are finite-dimensional, thus $B(\mathcal{H})$ is the set of all operators (complex matrices).

<u>Def.</u>: A <u>noisy operation</u> $\mathcal{E}: B(\mathcal{H}_{in}) \to B(\mathcal{H}_{out})$ is any map of the form

$$\mathcal{E}(\rho_{in}) = \mathrm{Tr}_{a'}\left[ U(\rho_{in} \otimes \gamma_a) U^\dagger \right]$$

where $U$ is any unitary on $\mathcal{H} = \mathcal{H}_{in} \otimes \mathcal{H}_a$, $\mathcal{H}_a$ is a finite-dimensional ("ancilla") Hilbert space, $\gamma_a = \mathbb{1}_a / d_a$, $d_a = \dim \mathcal{H}_a$, the maximally mixed state on $\mathcal{H}_a$, and $\mathcal{H} = \mathcal{H}_{out} \otimes \mathcal{H}_{a'}$.




These are the maps in a resource theory, where an agent is given $\rho_{in}$, and then he can

- introduce maximally mixed ("uniform") states $\gamma_a$ for free,
- apply reversible transformations $U$ for free,
- discard (trace out) subsystems.

Note that

$$\mathcal{E}(\gamma_{in}) = \mathcal{E}\left(\mathbb{1}/d_{in}\right) = \mathrm{Tr}_{a'}\left[U\left(\mathbb{1}/d_{in} \otimes \mathbb{1}/d_a\right)U^\dagger\right]$$

$$= \mathrm{Tr}_{a'}\left[\mathbb{1}/d_{out} \otimes \mathbb{1}/d_{a'}\right] = \mathbb{1}/d_{out} = \gamma_{out}.$$

Noisy operations preserve the maximally mixed state. Max. mixed states are "free"; all other states are valuable "resources".

Later in this lecture: every subsystem $S$ has its Hamiltonian $H_S$, and there is an extra condition on the allowed unitaries:

$$[U, H_{in} + H_a] = 0 \quad \text{(total energy conservation).}$$

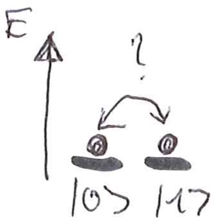
Also, thermal states $\gamma_a := \exp(-\beta H_a)/Z$ are free, not the maximally mixed states

→ **"resource theory of athermality"** at inverse temp. $\beta$.

Special case: $H_S = 0$ for all $S$ → resource theory of nonuniformity

## 2.3. Noisy classical operations

Recall Landauer erasure:




quantum state $\rho = \lambda_0 |\psi_0\rangle\langle\psi_0| + \lambda_1 |\psi_1\rangle\langle\psi_1|$ $\xrightarrow{\text{unitary } U}$ $\rho' = U\rho U^\dagger = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1| = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

$U$ is "free" (no work cost); $\rho'$ describes classical probability distr. over the two levels.

Recall some definitions and notation from classical probability theory:

A <u>discrete probability space</u> is a pair $(\Omega, P)$, where $\Omega$ is a finite set (of "outcomes"), and $P: \Omega \to [0,1]$ is a function such that $\sum_{\omega \in \Omega} P(\omega) = 1$.

$\Omega$ is also called the "sample space".

<u>Example</u>: a fair die: $\Omega = \{1, 2, \dots, 6\}$

$P(\omega) = \frac{1}{6}$ for all $\omega \in \Omega$.

Notation: $P = (\frac{1}{6}, \dots, \frac{1}{6}) \in \mathbb{R}^6$.

For prob. spaces $A = (\Omega_A, P_A)$, $B = (\Omega_B, P_B)$, a <u>composite</u> probability space $AB$ is

$$AB = (\Omega_A \times \Omega_B, P_{AB}),$$

where $P_{AB}$ has $P_A$ and $P_B$ as marginals (see below).

If $P_A \in \mathbb{R}^m$, $P_B \in \mathbb{R}^n$, then $P_{AB} \in \mathbb{R}^m \otimes \mathbb{R}^n \simeq \mathbb{R}^{mn}$.

$P_{AB}$ is <u>uncorrelated</u> if $P_{AB}(a, b) = P_A(a) \cdot P_B(b)$ for all $a \in \Omega_A$, $b \in \Omega_B$.

$\iff P_{AB} = P_A \otimes P_B$ because

$$P_A = (P_A(a_1), \dots, P_A(a_m)) \in \mathbb{R}^m, \quad P_B = (P_B(b_1), \dots, P_B(b_n)) \in \mathbb{R}^n,$$

$$P_A \otimes P_B = (P_A(a_1)P_B(b_1), P_A(a_1)P_B(b_2), \dots, P_A(a_m)P_B(b_n)) \in \mathbb{R}^{mn}.$$

The marginal distribution of $P_{AB}$ on $A$ is

$$P_A(a) = \sum_{b \in \Omega_B} P_{AB}(a, b) \quad \text{(and similarly for } P_B\text{)}.$$

Example: two (perfectly) correlated bits a, b
either $(a,b) = (0,0)$ or $(1,1)$ with 50% prob.

$\Rightarrow P_{AB}(0,0) = P_{AB}(1,1) = \frac{1}{2}$ ; $P_{AB}(0,1) = P_{AB}(1,0) = 0$.

$\Rightarrow P_A(0) = P_{AB}(0,0) + P_{AB}(0,1) = \frac{1}{2}$

$P_A(1) = P_{AB}(1,0) + P_{AB}(1,1) = \frac{1}{2}$

$\Rightarrow P_A = \left(\frac{1}{2}, \frac{1}{2}\right)$ maximally mixed

$P_{AB} \neq P_A \otimes P_B = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \Rightarrow$ correlation.

There is a strong formal and conceptual analogy between quantum states and classical "states" (probability distributions). Here is an incomplete "dictionary":

| Quantum notion | classical analog |
| --- | --- |
| density matrix $\rho$ | probability distribution $P$ |
| composite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ | Product sample space $\Omega_{AB} = \Omega_A \times \Omega_B$ |
| bipartite state $\rho_{AB}$ | bipartite prob. dist. $P_{AB}$ |
| local reduced state $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ | marginal $P_A(a) = \sum_{b\in\Omega_B} P_{AB}(a,b)$ |

| | |
|---|---|
| product state $\rho_A \otimes \rho_B$ | uncorrelated distr. $P_{AB} = P_A \otimes P_B$ |
| pure state $\lvert\psi\rangle\langle\psi\rvert$ | $P = (0, 0, \dots, 0, 1, 0, \dots, 0)$ |
| unitary map $\rho' = U \rho U^\dagger$ | Permutation $P' = \pi[P]$<br>$\pi : \Omega \to \Omega$ bijective, i.e.<br>$(P'(a_1), \dots, P'(a_m))$ [redacted]<br>$= (P(a_{\pi(1)}), \dots, P(a_{\pi(m)}))$ |
| von Neuman entropy<br>$S(\rho) = -\operatorname{tr}(\rho \log \rho)$;<br>$S(U\rho U^\dagger) = S(\rho)$ | Shannon entropy<br>$H(P) := -\sum_{i=1}^{m} P(a_i) \log P(a_i)$<br>$H(\pi[P]) = H(P)$. |

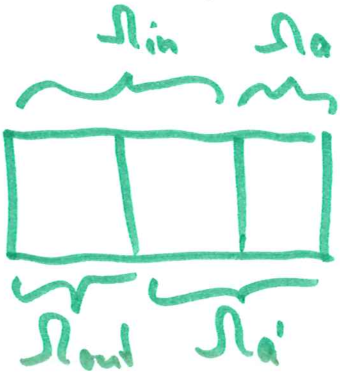
Note that $S(\rho) = H(\lambda(\rho))$, with $\lambda(\rho) = (\lambda_1, \dots, \lambda_m)$ the eigenvalues of $\rho$, i.e. $\rho = \sum_{i=1}^{m} \lambda_i \underbrace{\lvert i\rangle\langle i\rvert}_{\text{eigenbasis}} \in B(\mathbb{C}^m)$.

__Def.__: A map $D : \mathbb{R}^{d_{in}} \to \mathbb{R}^{d_{out}}$ is a __noisy classical operation__ if there exists a finite sample space $\Omega_a$ with uniform distribution $\gamma_a = \left(\frac{1}{|\Omega_a|}, \dots, \frac{1}{|\Omega_a|}\right)$ and a permutation $\pi$ with

$$P_{out} := D(P_{in}) = \left(\pi[P_{in} \otimes \gamma_a]\right)_{out}$$

marginal

i.e. $P_{out}(i) = \sum_{j=1}^{|\Omega_{a'}|} \left(\pi[P_{in} \otimes \gamma_a]\right)(i, j)$, where $\Omega_{in} \times \Omega_a = \Omega_{out} \times \Omega_{a'}$.




(This is just a sketch. It need not be that $\Omega_a \subset \Omega_{a'}$.)

Write $P \overset{\text{noisy}}{\Longrightarrow} Q$ for prob. distr. $P \in \mathbb{R}^m$, $Q \in \mathbb{R}^n$ if there exists a noisy classical operation $D$ with $Q = D(P)$.

<u>Example</u>: $P = (1,0) \in \mathbb{R}^2$ a "pure state" on $\Omega_{in} = \{1,2\}$.

$$\Rightarrow P \otimes \gamma_a = (1,0) \otimes \left(\frac{1}{|\Omega_a|}, \dots, \frac{1}{|\Omega_a|}\right) = \left(\frac{1}{|\Omega_a|}, \dots, \frac{1}{|\Omega_a|}, 0, \dots, 0\right).$$

$$\Rightarrow \pi[P \otimes \gamma_a] = (\tilde{p}_1, \dots, \tilde{p}_{2|\Omega_a|}), \text{ all } \tilde{p}_i \in \left\{0, \frac{1}{|\Omega_a|}\right\}.$$

(half of them zeroes)

Set $\Omega_{out} = \Omega_{in}$, $\Omega_{a'} = \Omega_a$. Then computing the marginal picks out $|\Omega_a|$ many entries and adds them up. Choose $\pi$ appropriately

$\Rightarrow$ we can get any $P_{out}$ of the form

$$P_{out}(1) = m \cdot \frac{1}{|\Omega_a|}, \text{ where } 0 \le m \le |\Omega_a|, \; m \in \mathbb{N}_0$$

$$P_{out}(2) = 1 - P_{out}(1).$$

Thus $(1,0) \overset{\text{noisy}}{\Longrightarrow} Q \iff Q = (q_1, q_2)$ with $q_1 + q_2 = 1$, $q_i \ge 0$, $q_i \in \mathbb{Q}$ (rational).

From a physics point of view, perfect generation of the target distribution is unnecessary.

Def.: We write $P \xrightarrow{\text{noisy}} Q$
if for every $\varepsilon > 0$ there exists $Q_\varepsilon$ with $\|Q - Q_\varepsilon\| < \varepsilon$ and a noisy classical operation $D_\varepsilon$ with $D_\varepsilon(P) = Q_\varepsilon$.

Lemma: Let $\rho, \sigma$ be quantum states. There exists a noisy (quantum) operation $\mathcal{E}$ with $\mathcal{E}(\rho) = \sigma$ if and only if

$$\underbrace{\lambda(\rho)}_{\text{eigenvalues}} \xrightarrow{\text{noisy}} \lambda(\sigma).$$

In effect, "everything becomes classical". (This will not be true any more for the "resource theory of athermality" later in the lecture.)

How can we decide whether $P \xrightarrow{\text{noisy}} Q$ ?

→ Majorization.

[redacted]

Lemma: Let $P, Q$ have the same size, i.e. $P, Q \in \mathbb{R}^n$.
Then $P \xrightarrow{\text{noisy}} Q$ if and only if $P$ majorizes $Q$, i.e. $P \succ Q$.

Def.: Let $P, Q \in \mathbb{R}^m$ be probability vectors. We write $P \succ Q$ ("$P$ majorizes $Q$") iff

$$\sum_{i=1}^{k} P_i^{\downarrow} \geq \sum_{i=1}^{k} Q_i^{\downarrow} \quad \text{for all } k = 1, \dots, m,$$

where $P_1^{\downarrow} \geq P_2^{\downarrow} \geq P_3^{\downarrow} \geq \dots$ denotes the entries of $P$ in decreasing order.

Example: $(1/5, 0, 4/5) \succ (1/3, 1/3, 1/3)$, because

- $k=1$: $4/5 \geq 1/3$ ✓
- $k=2$: $4/5 + 1/5 \geq 1/3 + 1/3$ ✓
- $k=3$: $4/5 + 1/5 + 0 \geq 1/3 + 1/3 + 1/3$ ✓

However, consider $P = (0.4, 0.4, 0.1, 0.1)$ and $Q = (0.5, 0.25, 0.25, 0)$. Then $P \not\succ Q$ and $Q \not\succ P$

$\Rightarrow$ neither one is a "better resource" than the other!

That is, **"$\succ$" is a preorder**, not a (total) order.

I.e. $P \succ P$, $P \succ Q$ and $Q \succ R \Rightarrow P \succ R$

but, for instance, $P \succ Q$ and $Q \succ P \not\Rightarrow P = Q$

and $\neg(P \succ Q) \not\Rightarrow Q \succ P$.