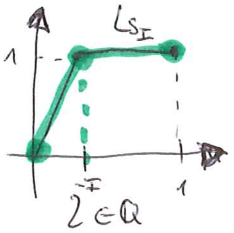
Single-shot thermo, lecture 6

25.11.2014

- There are now handwritten lecture notes online. → correction to lecture 2!
- Exercises 5: now 4 problems, but 2 weeks of time to solve them!

- Quick repetition from last time:

Sharp state s_I: if $I \in \mathbb{N}$, then $s_I = \binom{1}{0}^{\otimes I}$ ("I pure bits"); otherwise



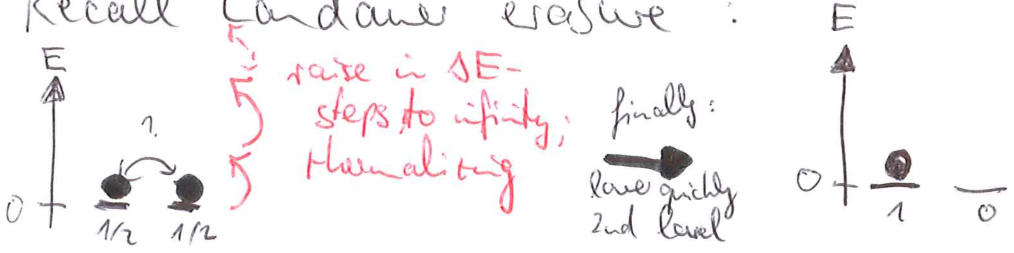
Distillable nonuniformity: $p \xrightarrow{\text{noisy}} s_I$ iff $I \leq I_0(p) := \log d_p - H_0(p)$

Nonuniformity of formation:

$$s_I \xrightarrow{\text{noisy}} p \quad \text{iff} \quad I \geq I_\infty(p) := \log d_p - H_\infty(p).$$

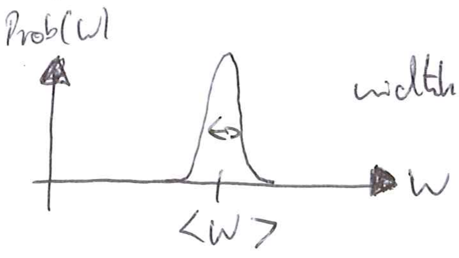
2.8. Interpreting I_∞ (and I_0): a clarification

Recall Landauer erasure:



Expected work $\langle W \rangle = k_B T \log 2$

Shown by Browne, Garner, Dahlsten, Vedral; arXiv: 1311.7612:



width of distribution $\sigma \leq \frac{1}{2} \Delta E \sqrt{N}$

N: # of raising steps;

$\Delta E \to 0$ faster than $N \to \infty$ can have $\sigma \to 0$.

1

$\Rightarrow$ investing $k_B T \log 2 + \delta$ of work, with $\delta > 0$ arbitrarily small, we can make almost 100% sure to reset the bit.

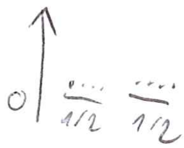
<u>Homework exercise 5:</u> Suppose we don't want $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ but $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1-p \\ p \end{pmatrix}$, $0 \leq p \leq \frac{1}{2}$. (don't raise E all up to infinity!)

Then
$$\langle W \rangle = k_B T \cdot \underbrace{I_\infty(p)}_{\text{"work of formation" of } p} = k_B T \left[\log 2 - H_\infty(p) \right]$$

and work distribution is still arbitrarily peaked!

So what about standard Shannon / von Neumann entropy?

If we have $n \in \mathbb{N}$ (n large) independent particles, distributed as $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}^{\otimes n}$



and we want to get the distributed close to $\begin{pmatrix} 1-p \\ p \end{pmatrix}^{\otimes n}$

Then $\langle W \rangle = k_B T \left(n \cdot H(p) + o(n) \right)$

(we will prove this soon).

Similar reasoning applies to Szilard engines and distillable work $I_0(p)$.

2.9. Approximate formation and distillation

Write $p \xrightarrow{\varepsilon\text{-noisy}} q$ (where $\varepsilon \geq 0$) if there exists a prob. vector $\tilde{q}$ with $D(q, \tilde{q}) \leq \varepsilon$ and $p \xrightarrow{\text{noisy}} q$.

trace distance / variation distance

Consistency of quantum and classical cases:

<u>Lemma</u>: For quantum states ρ, σ, there exists a noisy (quantum) operation $\mathcal{N}$ and another state $\tilde{\sigma}$ with $D(\sigma, \tilde{\sigma}) \leq \varepsilon$ and $\mathcal{N}(\rho) = \tilde{\sigma}$ if and only if $\lambda(\rho) \xrightarrow{\varepsilon\text{-noisy}} \lambda(\sigma)$.

Proof not here.

<u>Definition</u> ("smooth entropies")

For a prob. vector p and $\varepsilon > 0$, define

$$H_\infty^\varepsilon(p) := \max_{p': D(p,p') \leq \varepsilon} H_\infty(p'), \text{ and}$$

$$H_0^\varepsilon(p) := \min_{p': D(p,p') \leq \varepsilon} H_0(p').$$

Smooth entropies (conditional versions etc.) have <u>many</u> applications in classical and quantum information theory.

Non-uniformity of approximate formation:

<u>Lemma</u>: The minimum I with $s_I \xrightarrow{\varepsilon\text{-noisy}} p$ is given by

$$I_\infty^\varepsilon(p) := \log d_p - H_\infty^\varepsilon(p).$$

<u>Proof</u>: Suppose $s_I \xrightarrow{\text{noisy}} p'$. Minimal I needed for this is

$$I_\infty(p') = \log \underbrace{d_p}_{= d_{p'}} - H_\infty(p').$$

Minimize this over all p' with $D(p, p') \leq \varepsilon$:

3

$$\min_{p': D(p,p') \le \varepsilon} I_\infty(p') = \log d_p - \max_{p': D(p,p') \le \varepsilon} H_\infty(p')$$

$$= \log d_p - H_\infty^\varepsilon(p). \qquad \square$$

<u>Interpretation</u>: $k_B T\, I_\infty^\varepsilon(p)$ is the amount of work we have to invest to create p approximately / with failure probability $\le \varepsilon$. (cf. explanations in 2.7 / interpretation of trace distance.)

Approximate [redacted] distillable nonuniformity:

<u>Lemma</u>: The maximum I with $p \xrightarrow{\varepsilon\text{-noisy}} S_I$ is <u>at least</u>

$$I_0^\varepsilon(p) := \log d_p - H_0^\varepsilon(p).$$

(Exact value: later.)

<u>Proof</u>: Choose p' with $D(p,p') \le \varepsilon$ and $H_0^\varepsilon(p) = H_0(p')$.

Then there exists a noisy operation $\mathcal{N}$ such that

$$\mathcal{N}(p') = S_I, \text{ where } I = I_0(p') = I_0^\varepsilon(p).$$

Apply $\mathcal{N}$ to p instead, and set $q := \mathcal{N}(p)$, we get

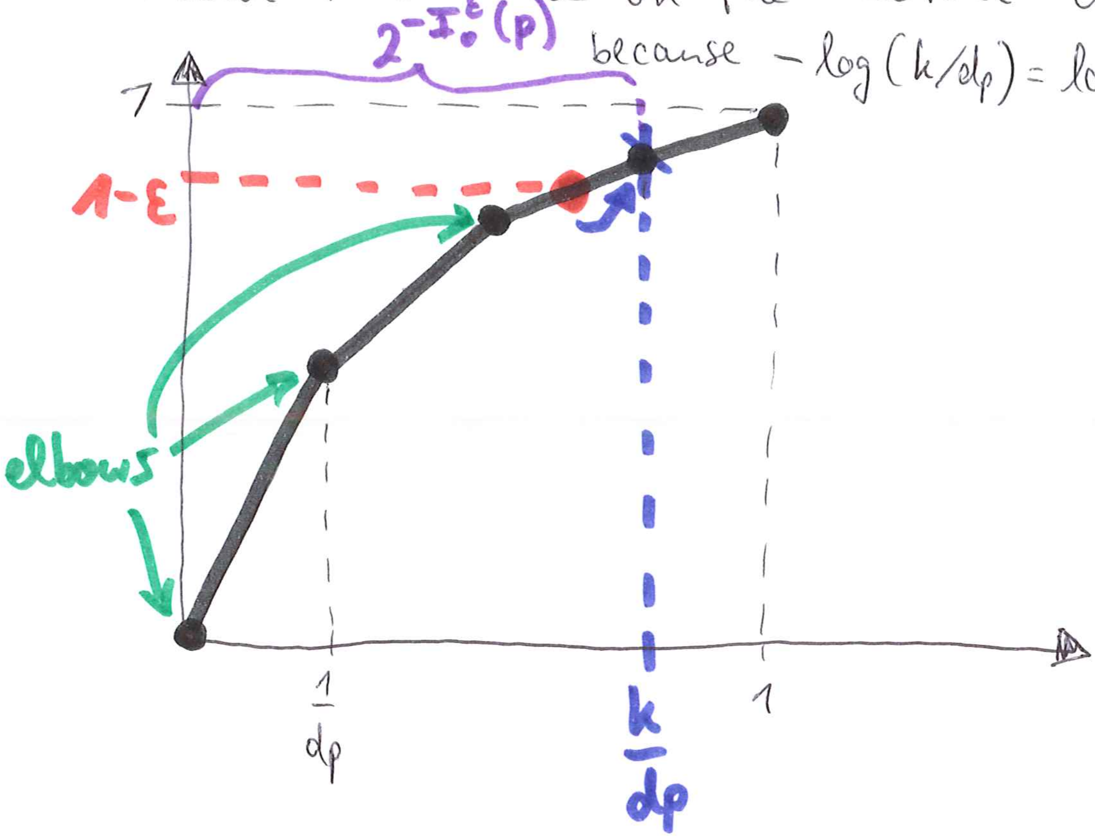
$$D(q, S_I) = D(\mathcal{N}(p), \mathcal{N}(p')) \le D(p,p') \le \varepsilon.$$

Thus $p \xrightarrow{\varepsilon\text{-noisy}} S_I$ for $I = I_0^\varepsilon(p)$. $\square$

<u>Homework</u>: $H_0^\varepsilon(p) = \log k$, where $k \ge 1$ is the smallest integer with $\sum_{i=1}^{k} x_i^\downarrow \ge 1 - \varepsilon$.

Find this value on the Lorenz curve:

because $-\log(k/d_p) = \log d_p - H_0^\varepsilon(p) = I_0^\varepsilon(p)$



<u>Lemma</u>: The maximum I with $p \xrightarrow{\varepsilon\text{-noisy}} S_I$ is exactly

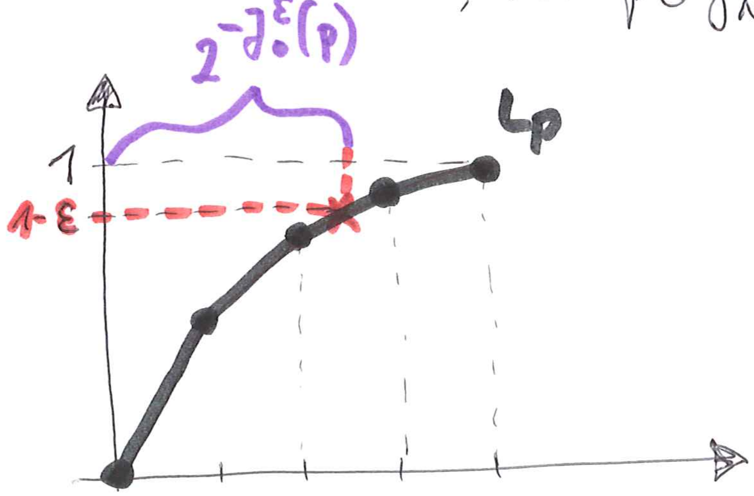
$$J_0^\varepsilon(p) := \lim_{N\to\infty} I_0^\varepsilon(p \otimes \gamma_N)$$

$\left(\frac{1}{N}, \dots, \frac{1}{N}\right)$

"$\geq$" is clear because p and $p\otimes\gamma_N$ are "noisy-equivalent";
"$\leq$" needs more work to prove (not here).

p and $p\otimes\gamma_N$: Same Lorenz curves, but $p\otimes\gamma_N$ has "more **elbows**"

$\Rightarrow J_0^\varepsilon$ easy to read off:



Discuss why.

5

2.10. State conversion in the thermodynamic limit: recovering Shannon and von Neumann entropy

Thermodynamic limit: we have n systems/particles that are not strongly correlated, and n is large. Ideally, we have $p^{\otimes n}$, and send $n \to \infty$.

Lemma (asymptotic equipartition property):
For every $0 < \varepsilon < 1$, we have

$$\lim_{n\to\infty} \frac{1}{n} H_0^{\varepsilon}(p^{\otimes n}) = H(p), \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} H_{\infty}^{\varepsilon}(p^{\otimes n}) = H(p),$$

where H is Shannon entropy, $H(p) = -\sum_i p_i \log p_i$.

Proof: Next time. Follows from Shannon's noiseless coding theorem (data compression!).

Homework: you show the following: leave on blackboard.

$$\boxed{\text{Lemma: If } I_0^{\varepsilon/2}(p) \geq I_{\infty}^{\varepsilon/2}(q) \text{ then } p \xrightarrow{\varepsilon\text{-noisy}} q.}$$

We will use this to answer the following

Questions:
- Suppose we have many copies of p. How many copies of q can we distill?
- Suppose we want many copies of q. How many copies of p do we need?

<u>Theorem</u>: p, q prob. vectors, not [redacted] maximally mixed, $0 < \varepsilon < 1$. For every $n \in \mathbb{N}$, let m_n be the largest integer such that

$$p^{\otimes n} \xrightarrow{\varepsilon\text{-noisy}} q^{\otimes m_n}.$$

Then
$$\lim_{n\to\infty} \frac{m_n}{n} = \frac{I(p)}{I(q)}$$

with $I(p) := \log d_p - H(p)$ the (Shannon) negentropy.
Similarly, let k_n be the smallest integer such that

$$p^{\otimes k_n} \xrightarrow{\varepsilon\text{-noisy}} q^{\otimes n}.$$

Then
$$\lim_{n\to\infty} \frac{k_n}{n} = \frac{I(q)}{I(p)}.$$

Interpretation:

- In the thermodynamic limit, negentropy I is the "unique measure of information".
- "Distillable nonuniformity": $p^{\otimes n} \xrightarrow{\varepsilon\text{-noisy}} \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{pure}}^{\otimes m_n}$

  works for $m_n = n \cdot I(p) + o(n)$

  $\Rightarrow$ get $\approx n \cdot I(p)$ pure bits from $p^{\otimes n}$.
- "Nonuniformity of formation": $\begin{pmatrix}1\\0\end{pmatrix}^{\otimes k_n} \xrightarrow{\varepsilon\text{-noisy}} p^{\otimes n}$

  works for $k_n = n \cdot I(p) + o(n)$

  $\Rightarrow$ need $\approx n \cdot I(p)$ pure bits to create $p^{\otimes n}$.

$\Rightarrow$ Noisy operations become "reversible" in the thermodynamic limit!

However, in the microscopic / single-shot regime, we have <u>irreversibility</u> due to $\underbrace{I_0^\varepsilon(p)}_{\text{distillable}} \ll \underbrace{I_\infty^\varepsilon(p)}_{\text{formation}}$ in general.

<u>Proof of theorem</u>: Lemma on BB $\Rightarrow$

if $\frac{m}{n} \leq \dfrac{\frac{1}{n} I_0^{\varepsilon/2}(p^{\otimes n})}{\frac{1}{m} I_\infty^{\varepsilon/2}(q^{\otimes m})}$ then $p^{\otimes n} \xrightarrow{\varepsilon\text{-noisy}} q^{\otimes m}$

But this is impossible for $m = m_n + 1$

$$\Rightarrow \quad \frac{m_n+1}{n} > \frac{\frac{1}{n} I_0^{\varepsilon/2}(p^{\otimes n})}{\text{[redacted]}\ \frac{1}{m_n+1} I_\infty^{\varepsilon/2}(q^{\otimes(m_n+1)})}$$

Easy to see: [redacted] $\lim_{n\to\infty} m_n = \infty$, so

$$\liminf_{n\to\infty} \frac{m_n}{n} \geq \liminf_{n\to\infty} \frac{\frac{1}{n} I_0^{\varepsilon/2}(p^{\otimes n})}{\frac{1}{m_n+1} I_\infty^{\varepsilon/2}(q^{\otimes(m_n+1)})}$$

$$= \frac{\log d_p - \lim_{n\to\infty} \frac{1}{n} H_0^{\varepsilon/2}(p^{\otimes n})}{\log d_q - \lim_{m\to\infty} \frac{1}{m} H_\infty^{\varepsilon/2}(q^{\otimes m})} = \frac{I(p)}{I(q)}.$$

We have $p^{\otimes n} \xrightarrow{\text{noisy}} q_n$ with $D(q^{\otimes m_n}, q_n) \leq \varepsilon$.

Choose $\delta > 0$ such that $\varepsilon + \delta < 1$.

Homework: I_∞^δ is a nonuniformity monotone $\Rightarrow$

$$I_\infty^\delta(p^{\otimes n}) \geq I_\infty^\delta(q_n) = I_\infty(q_n')$$

with $D(q_n, q_n') \leq \delta$.

$$\Rightarrow D(q^{\otimes m_n}, q_n') \leq D(q^{\otimes m_n}, q_n) + D(q_n, q_n') \leq \varepsilon + \delta$$

$$\Rightarrow I_\infty^{\delta+\varepsilon}(q^{\otimes m_n}) = \min_{q' : D(q^{\otimes m_n}, q') \leq \delta + \varepsilon} I_\infty(q')$$

$$\leq I_\infty(q_n') = I_\infty^\delta(q_n) \leq I_\infty^\delta(p^{\otimes n}).$$

$$\Rightarrow \frac{m_n}{n} \leq \frac{\frac{1}{n} I_\infty^\delta(p^{\otimes n})}{\frac{1}{m_n} I_\infty^{\varepsilon+\delta}(q^{\otimes m_n})}$$

$$\Rightarrow \limsup_{n\to\infty} \frac{m_n}{n} \leq \frac{\lim_{n\to\infty} \frac{1}{n} I_\infty^\delta(p^{\otimes n})}{\lim_{n\to\infty} \frac{1}{m_n} I_\infty^{\varepsilon+\delta}(q^{\otimes m_n})} = \frac{I(p)}{I(q)}.$$

Second statement can be proven analogously. $\square$