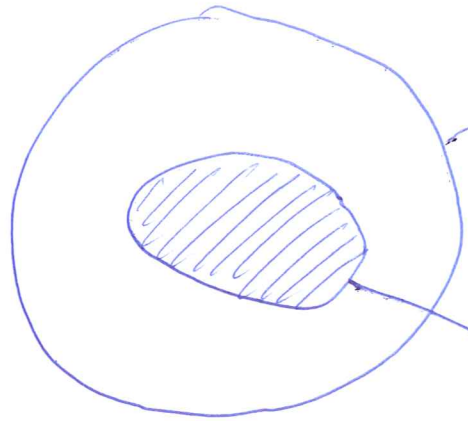


# Single-shot thermo, lecture 8

10.12.2014

Last time, we talked about the typical subset theorem / Shannon's noiseless coding theorem.

$X_i \in X$  random variables,  $X_1, X_2, X_3, \dots$  iid  $\sim p(x)$ .



all  $|X|^n$  possible outcome sequences

"typical" set  $A_\epsilon^{(n)}$   
 $\approx 2^{nH}$  elements  
 $P(A_\epsilon^{(n)}) \approx 1$ .

→ data compression

→ AEP:  $\lim_{n \rightarrow \infty} \frac{1}{n} \log_{10} \epsilon (p^{\otimes n}) = H(p)$ .

Any questions on this?

## 2.11. Catalysis

... was first discovered in entanglement theory (more later on).

Example: Consider the two states

$$p = \left( \frac{2}{5}, \frac{2}{5}, \frac{1}{10}, \frac{1}{10} \right), \quad q = \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0 \right).$$

$$q_1^\downarrow > p_1^\downarrow \Rightarrow p \nprec q$$

$$p_1^\downarrow + p_2^\downarrow > q_1^\downarrow + q_2^\downarrow \Rightarrow q \nprec p$$

No state majorizes the other.

But consider  $C = (\frac{3}{5}, \frac{2}{5})$  and

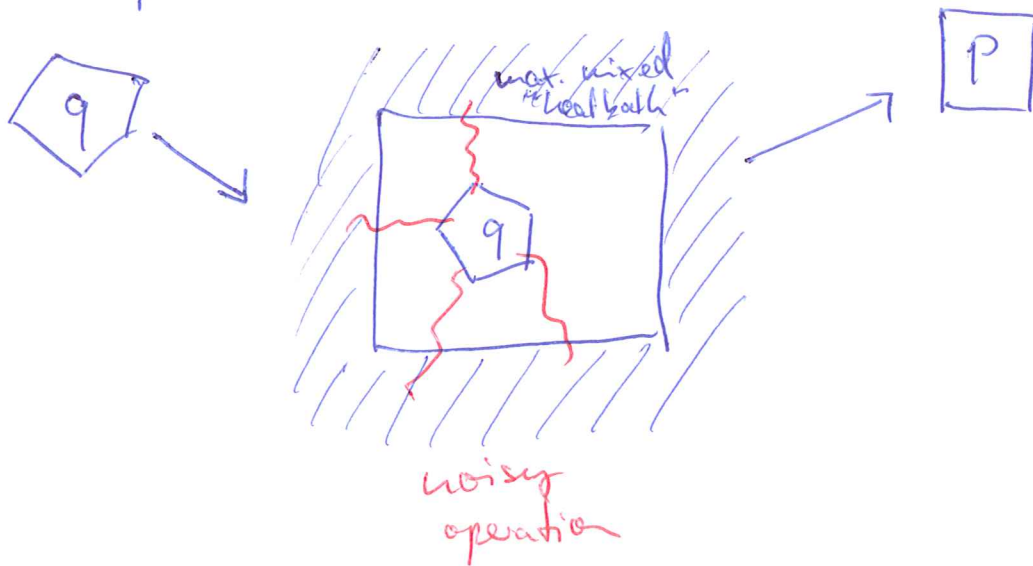
$$p \otimes C = \left( \frac{2}{5} \cdot \frac{3}{5}, \frac{2}{5} \cdot \frac{2}{5}, \frac{1}{5} \cdot \frac{3}{5}, \frac{1}{5} \cdot \frac{2}{5}, \frac{2}{5} \cdot \frac{3}{5}, \frac{2}{5} \cdot \frac{2}{5}, \frac{1}{5} \cdot \frac{3}{5}, \frac{1}{5} \cdot \frac{2}{5} \right) \in \mathbb{R}^8$$

and analogously  $q \otimes C$ .

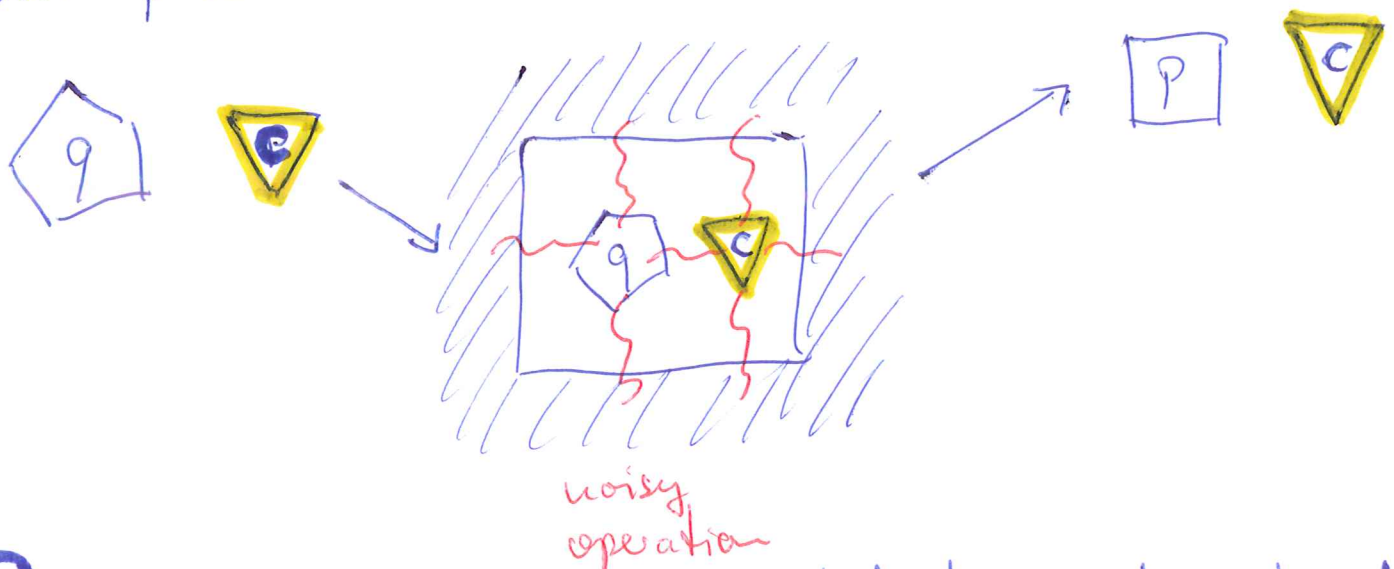
Then it turns out that  $q \otimes C > p \otimes C$

Thus  $q \xrightarrow{\text{noisy}} p$  but  $q \otimes C \xrightarrow{\text{noisy}} p \otimes C$ .

Impossible:



but possible:



②  $c$  can be reused - like a catalyst in chemistry!

Def. Let  $p, q \in \mathbb{R}^m$  be probability vectors.  
We say that  $p$  trumps  $q$ , and write

$$p \succ_T q$$

if there exists a (finite) probability vector  $c$  (a "catalyst")  
such that  $p \otimes c \succ q \otimes c$ .

If  $p \succ q$  then  $p \succ_T q$  (choose  $c = (1) \in \mathbb{R}^1$ ).

In fact,  $p \succ q \Rightarrow p \otimes c \succ q \otimes c$  for all  $c$   
(proof: exercise).

Example above shows: there are  $p, q$  with  $p \not\succ q$   
but  $p \succ_T q$ .

Lemma: Trumping is transitive:

$$p \succ_T q \text{ and } q \succ_T r \Rightarrow p \succ_T r.$$

Proof:  $\exists c: p \otimes c \succ q \otimes c$

$$\exists c': q \otimes c' \succ r \otimes c'$$

$$\Rightarrow p \otimes c \otimes c' \succ q \otimes c \otimes c'$$

$$q \otimes c \otimes c' \succ r \otimes c \otimes c'$$

$$\Rightarrow p \otimes (c \otimes c') \succ r \otimes (c \otimes c'). \quad \square$$

We can decide majorization via the Lorenz curve.  
How can we decide trumping?



## Theorem (Klimesh, Turgut 2007)

Let  $p, q \in \mathbb{R}^n$  be probability vectors with  $p^\downarrow \neq q^\downarrow$ .

Then  $p \succeq_T q$  if and only if

$$H_\alpha(p) < H_\alpha(q) \quad \forall \alpha \in \mathbb{R} \setminus \{0\}, \text{ and}$$

$$H_{\text{Burg}}(p) < H_{\text{Burg}}(q).$$

One proof direction simple  
( $H_\alpha(p \otimes c) = H_\alpha(p) + H_\alpha(c)$ ), the  
other very difficult.

$H_{\text{Burg}}$  is the "Burg entropy"

$$H_{\text{Burg}}(p) := \sum_{i=1}^n \log p_i.$$

$H_\alpha$  are the Rényi entropies,

$$H_\alpha(p) := \frac{\text{sgn}(\alpha)}{1-\alpha} \log \sum_{i=1}^n p_i^\alpha \quad (\alpha \in \mathbb{R} \setminus \{0, 1\}),$$

$$H_1(p) \equiv H(p) = - \sum_{i=1}^n p_i \log p_i.$$

Physical meaning: will be discussed later on, when we also model Hamiltonians / energy.

For now:

Standard second law in the thermodynamic limit:  
entropy  $H$  cannot decrease.

Now we have an infinite family of "second laws",  
the Rényi entropies all cannot decrease.

Lemma: We can always assume that catalysts have full rank.  
In particular, pure states are useless as catalysts.

Proof: Suppose  $p \otimes c > q \otimes c$ ,  $c \in \mathbb{R}^n$ ,  $c_\bullet = (c_1, c_2, \dots, c_{n-1}, 0)$   
 $= \tilde{c} \oplus 0$   
 $\Rightarrow p \otimes c = p \otimes (\tilde{c} \oplus 0) = p \otimes \tilde{c} \oplus \underbrace{(0, 0, \dots, 0)}_n$

$$\text{also } q \otimes c = q \otimes \tilde{c} \oplus \underbrace{(0, 0, \dots, 0)}_n$$

$\Rightarrow p \otimes \tilde{c} \succ q \otimes \tilde{c}$ ; we  
can remove zeroes subsequently from  $c$ .  $\square$

Catalysis does not increase the distillable nonuniformity (extractable work):

$$p \otimes c \xrightarrow{\text{noisy}} S_I \otimes c \quad \Rightarrow \quad p \otimes c \xrightarrow{\text{noisy}} S_I$$

$\underbrace{\quad}_{\text{sharp state}}$  Can choose  $c$  to have full rank.

The maximal  $I$  for this is

$$\begin{aligned} I_0(p \otimes c) &= \log d_{p \otimes c} - H_0(p \otimes c) = \log d_p + \log d_c - H_0(p) \\ &\quad - \underbrace{H_0(c)}_{\log \text{rank } c = \log d_c} \\ &= \log d_p - H_0(p) = I_0(p) \end{aligned}$$

and this is the maximal  $I$  with  $p \xrightarrow{\text{noisy}} S_I$ .

Catalysis does not decrease the nonuniformity of formation:

Use the fact that  $I_\alpha(p) := \log d_p - H_\alpha(p)$  ( $\alpha \geq 0$ ) is a nonuniformity monotone;

$$S_I \otimes c \xrightarrow{\text{noisy}} p \otimes c \Rightarrow I_\alpha(S_I \otimes c) \geq I_\alpha(p \otimes c) \quad \forall \alpha$$

Easy to check:  $I_\alpha$  additive on tensor products

$$\Rightarrow I_\alpha(S_I) + I_\alpha(c) \geq I_\alpha(p) + I_\alpha(c)$$

Take limit  $\alpha \rightarrow \infty$  and use  $I_\alpha(S_I) = I \quad \forall \alpha$ :

$$I \geq I_\infty(p)$$

But  $I_\infty(p)$  is the smallest  $I$  such that  $S_I \xrightarrow{\text{noisy}} p$ .

Similarly, one can show  $(*) S_I \otimes c \xrightarrow{\text{noisy}} p' \otimes c$  with  $D(p, p') \leq \epsilon$

  $\Leftrightarrow S_I \xrightarrow{\epsilon\text{-noisy}} p$

also nonuniformity of approximate formation remains the same.

However, distillable approx. uniformity can slightly increase for  $\epsilon > 0$  (open by how much).

## The embezzling phenomenon

$(*)$  insists on retrieving the catalyst perfectly intact. Why not allowing "approximate" catalysis?

For example

$$p \otimes c \xrightarrow{\epsilon\text{-noisy}} q \otimes c ? \quad (**)$$

Theorem: For every  $\epsilon > 0$ , there exists a catalyst  $c$  such that  $c \xrightarrow{\epsilon\text{-noisy}} S_1 \otimes c$ , where  $S_1 = (1, 0)$  is one pure bit.

We can "embezzle" (enterschlagen) a pure bit from the catalyst, while almost keeping it intact.

By repeating this (with  $\epsilon$  small enough), we can create any state we want  $\rightarrow$  not a good model of work extraction!

⑥



Proof: For  $n \in \mathbb{N}$ , set  $S_n := \sum_{i=1}^n \frac{1}{i}$  and

$$c^{(n)} := \frac{1}{S_n} \left( 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \right). \quad \text{Assume } n=2k, \text{ i.e. } n \text{ even.}$$

Example:  $n=4$  ( $k=2$ ).

$$\Rightarrow p := \left( c^{(4)} \otimes \left( \frac{1}{2}, \frac{1}{2} \right) \right)^\downarrow = \frac{1}{S_4} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right) \left| \begin{array}{cc} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{8} \end{array} \right|$$

$$q := \left( c^{(4)} \otimes (1, 0) \right)^\downarrow = \frac{1}{S_4} \left( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right) \left| \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right|$$

$$\Rightarrow D(p, q) = \frac{1}{2} \cdot \sum_{i=1}^k \frac{1}{S_n} \left| \frac{1}{2i} - \frac{1}{2i-1} \right| + \frac{1}{2} \sum_{i=k+1}^{2k} \frac{1}{S_n} \left| \frac{1}{2i} - 0 \right|$$

$$= \frac{1}{2 S_{2k}} \sum_{i=1}^k \left( \frac{1}{2i-1} - \frac{1}{2i} \right) + \frac{1}{4 S_{2k}} \sum_{i=k+1}^{2k} \frac{1}{i}$$

$$= \frac{1}{4 S_{2k}} \underbrace{\sum_{i=1}^k \frac{1}{i(2i-1)}}_{\text{converges for } k \rightarrow \infty} + \frac{S_{2k} - S_k}{4 S_{2k}}$$

For  $k \rightarrow \infty$  we have  $S_k = \log k + \gamma + o(k)$

$\gamma = 0.577 \dots$  Euler-Mascheroni constant

$$\frac{S_{2k} - S_k}{4 S_{2k}} \sim \frac{\log(2k) + \gamma - \log k - \gamma}{4 \log(2k) + 4\gamma} \sim \frac{\log 2}{4 \log(2k)} \xrightarrow{k \rightarrow \infty} 0$$

$\Rightarrow D(p, q) < \varepsilon$  if  $k$  large enough.

$$\Rightarrow p \xrightarrow[\text{via identity map}]{\varepsilon\text{-close}} q, \quad \text{i.e.} \quad c^{(n)} \otimes \left( \frac{1}{2}, \frac{1}{2} \right) \xrightarrow[\text{via some permutation}]{\varepsilon\text{-close}} c^{(n)} \otimes (1, 0).$$

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Discuss Fannes inequality:  $|H(p) - H(q)| \leq D \log(n-1) + H[(D, 1-D)]$   
 $D = D(p, q)$  variation distance