Single-shot thermo, lecture 10

- Please hand in your exercises right <u>now</u>.
- Exercises 8 will be online tomorrow.
- 10 exercise sheets in total. Need 60% of points.

<u>Quick recap last lecture</u>:

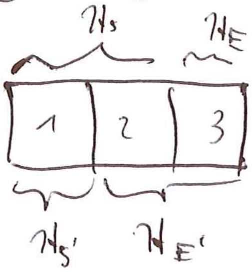
Resource theory of athermality (at fixed inv. temp. $\beta = \frac{1}{k_B T}$):

Thermal operations

$$\mathcal{E}(\rho_S) = \mathrm{Tr}_{E'}\left[U(\rho_S \otimes \gamma_E)U^\dagger\right]$$

where $[U, H_{SE}] = [U, H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E] = 0$

$$\gamma_E = \exp(-\beta H_E)/Z.$$



(Only the) Gibbs states γ_E are "free" (at inv. temp. β).

Motivation: these are the completely passive states

Reduces to <u>resource theory of nonuniformity</u> if all Hamiltonians are zero, i.e. $H_S = H_E = 0$.

<u>Today</u>:
- conversion rate is the <u>free energy</u> (thermodyn. limit)
- reduction to classical theory? (not quite).
- d-majorization

①

3.4. Transition rates in the thermodynamic limit: recovering the free energy

For all of the following, fix some inverse temperature $\beta \geq 0$!

A: system with state ρ_A, Hamiltonian H_A, i.e. $A = (\rho_A, H_A)$.

n copies $A^{\otimes n}$: system with state $\rho_A^{\otimes n}$, Hamiltonian $\underbrace{H_A + \ldots + H_A}_{n}$

We write $A \xrightarrow{\delta\text{-thermal}} B$ if there is a thermal operation $\mathcal{E}$ with $D(\mathcal{E}(\rho_A), \rho_B) \leq \delta$, and the quantum systems carry Hamiltonians H_A and H_B.

Fix $0 < \delta < 1$. For given A, B, let m_n be the largest integer with

$$A^{\otimes n} \xrightarrow{\delta\text{-thermal}} B^{\otimes m_n}$$

Suppose $H_A = H_B$ (special case, not really necessary). Then:

Theorem: $$\lim_{n\to\infty} \frac{m_n}{n} = \frac{D(\rho_A \| \gamma)}{D(\rho_B \| \gamma)} = \frac{F_\beta(\rho_A) - F_\beta(\gamma)}{F_\beta(\rho_B) - F_\beta(\gamma)},$$

roughly, a distance measure

where $D(\rho \| \sigma) = \mathrm{tr}(\rho \log \rho - \rho \log \sigma)$ is the relative entropy, and $D(\rho \| \gamma) = \beta F_\beta(\rho) - \beta F_\beta(\gamma)$ with $F_\beta(\rho) = \mathrm{tr}(H\rho) - S(\rho)/\beta$ the free energy.

cf. Exercises 1.

Proof sketch for special case $H_A = H_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$ and $\rho_B = |1\rangle\langle 1|$ ($\Rightarrow$ we want to get as many pure excited states as possible: work extraction!) as well as $[H_A, \rho_A] = 0$ (quasiclassical case).

$$\rho_A = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|,$$

$$\gamma = (1-p)|0\rangle\langle 0| + q|1\rangle\langle 1|, \quad q = e^{-\beta}/(1+e^{-\beta})$$

<u>We want:</u> $\gamma^{\otimes l} \otimes \rho_A^{\otimes n} \longrightarrow \sigma^{(k)} \otimes |1\rangle\langle 1|^{\otimes m} \quad (l+n=k+m)$

by some energy-preserving unitary, with m as large as possible.

Everything commutes $\to$ use classical typical subset theorem:

$\rho_A^{\otimes n} \approx$ uniform distribution on those strings of length n with $\approx np$ ones and $\approx n(1-p)$ zeroes

$\gamma^{\otimes l} \approx \ldots$ with $\approx lq$ ones and $\approx l(1-q)$ zeroes

$$\boxed{\text{This is just a sloppy proof sketch. More details: arXiv:1111.3882.}}$$

$$\underbrace{\overbrace{000\ldots0\underbrace{11\ldots1}_{lq}}^{l}}_{\text{in some order}}\ \underbrace{\overbrace{000\ldots0\underbrace{11\ldots1}_{np}}^{n}}_{\text{in some order}}$$

$\approx 2^{l\cdot h(q)} \times 2^{n\cdot h(p)}$ such strings

$$\xrightarrow[\text{unitary = permutation}]{\text{energy-preserving}} \underbrace{\overbrace{000\ldots0\underbrace{111\ldots11}_{rk}}^{k}}_{\text{in some order}}\ \overbrace{111\ldots1}^{m}$$

$\approx 2^{k\cdot h(r)}$ such strings

where $h(p) := H(p, 1-p) = -p\log p - (1-p)\log(1-p)$

binary Shannon entropy.

3 Several conditions must be satisfied:

• Unitarity = reversibility: $\Rightarrow 2^{k h(r)} \geq 2^{\ell h(q)} 2^{n h(p)}$

We will send $n \to \infty$, and also $\ell \to \infty$, and can have $\varepsilon = \frac{n}{\ell} \to 0$ (i.e. use many Gibbs states γ).

• Energy conservation: here, total energy $\simeq$ # of ones (cf. homework problem 23).

$$\ell q + n p = r k + m$$

• Same dimension: $\ell + n = k + m$ rate: $R = \frac{m}{n}$

Some little algebra $\Rightarrow$

$$h(q) + \varepsilon h(p) \leq (1 + \varepsilon - R\varepsilon)\, h\left(\frac{q + \varepsilon p - \varepsilon R}{1 + \varepsilon - \varepsilon R}\right)$$

Expand in first order in ε

$$\Rightarrow R \leq \frac{h(q) - h(p) + \beta(p-q)}{h(q) + \beta(1-q)} = \frac{S(\rho \| \gamma)}{S(|1\rangle\langle 1| \,\|\, \gamma)} \qquad \square$$

Special case: $H_A = H_B = 0$ $\rightarrow$ resource theory of nonuniformity.

$$D(\rho_A \| \gamma) = D(\rho_A \| \mathbb{1}/d_A) = \mathrm{tr}(\rho_A \log \rho_A - \rho_A \log \mathbb{1}/d_A)$$

$$= \log d_A - S(\rho_A) = I(\rho_A)$$

von Neumann / Shannon negentropy

Recovers the earlier result!

3.5. From the quantum to the classical case?

Recall resource theory of nonuniformity. We had:

$$\rho \xrightarrow{\text{noisy}} \sigma \text{ quantumly} \iff \lambda(\rho) \xrightarrow{\text{noisy}} \lambda(\sigma) \text{ classically} \qquad (*)$$

This is also reflected by the following equivalences:

Classically, equivalent are (for $p, q \in \mathbb{R}^n$ prob. vectors):

- $p \succ q$,
- there is a prob. [redacted] distr. $(r_1, \ldots, r_N)$ and permutations P_i such that $q = \sum_i r_i P_i p$
- there is a classical "channel" (stochastic matrix) B with $B \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix} = \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix}$ (i.e. B bistochastic) and $q = Bp$.
- For every $\varepsilon > 0$ there is q_ε with $\| q - q_\varepsilon \| < \varepsilon$ and [redacted] a noisy operation $\mathcal{N}_\varepsilon$ with $\mathcal{N}_\varepsilon(p) = q_\varepsilon$.

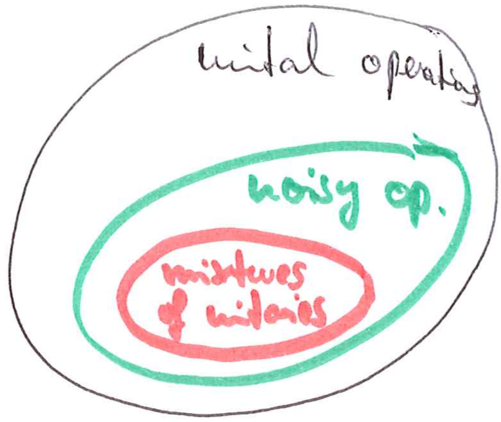
Quantumly, equivalent are (for ρ, σ states on $\mathbb{C}^n$):

- $\rho \succ \sigma$,
- there is a prob. distr. $(r_1, \ldots, r_N)$ and unitaries U_i such that $\sigma = \sum_i r_i U_i \rho U_i^\dagger$,
- (!) there is a quantum channel (CPTP map) ϕ with

$\phi(\mathbb{1}/n) = \mathbb{1}/n$ (i.e. ϕ unital) and $\sigma = \phi(\rho)$.

(!!) There is a noisy quantum operation $\mathcal{N}$ with $\mathcal{N}(\rho) = \sigma$.

Be careful: The sets of transformations are strictly disjoint, only the transitions that they allow are identical!



Classically the three sets of maps agree, but not quantumly (cf. Exercise 2, Birkhoff - von Neumann - Thm)

Unfortunately, **there is no analog of (*) for the resource theory of athermality!**

Cannot "diagonalize all states" – only U with $[U,H]=0$ are possible!

Homework Problem 28: thermal operations map diagonal states to diagonal states $\rightarrow$ no coherences can be created.

Def.: A state ρ is called (block-)diagonal (in the energy eigenbasis) if $[\rho, H] = 0$, i.e. in

Ex.: $H = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \Rightarrow \rho = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ 0 & 0 & \cdot \end{pmatrix}$ if $[\rho, H] = 0$.

Homework: If σ is diagonal, and T a thermal operation with $T(\rho)=\sigma$, then $T(\Delta(\rho))=\sigma$, where $\Delta(\rho)$ is [redacted] ρ with all off-diag. elements removed

($\Rightarrow$ in 3.4, condition $[H_A, \rho_A]=0$ was no restriction.)

Even the analog of (!) $\Leftrightarrow$ (!!) is wrong:

$$\boxed{\rho \to \sigma \text{ by a Gibbs-preserving map } \Phi}$$
$$\Uparrow \quad \not\Downarrow$$
$$\boxed{\rho \to \sigma \text{ by a thermal op. } T} \qquad \text{(homework).}$$

Open research problem (as of 2015):
Classify the set of all transitions $\rho \xrightarrow{\text{thermal}} \sigma$, for arbitrary (non-diagonal) ρ, σ.

In light of these problems, let's stick to block-diagonal states $\to$ everything becomes classical.

Classical system: $A=(p_A, H_A)$

$p_A \in \mathbb{R}^n$ prob. vector, $H_A=(E_1,\ldots,E_n)\in\mathbb{R}^n$ energies

Permutation $\pi:\mathbb{R}^n\to\mathbb{R}^n$ <u>energy-preserving</u> if $\pi[H_A]=H_A$.

Def. A map $D: \mathbb{R}^{d_S} \to \mathbb{R}^{d_{S'}}$ is a *thermal classical operation* if there is a system $E = (\gamma_E, H_E)$ with $(\gamma_E)_i = \exp(-\beta (H_E)_i)/Z$, and an energy-preserving permutation π on SE with

$$D(p_S) = \left(\pi\left[p_S \otimes \gamma_E\right]\right)_{S'} .$$

Write $p \xrightarrow{\text{thermal}} q$ if for every $\varepsilon > 0$ there is q_ε with $\|q - q_\varepsilon\| < \varepsilon$ and a thermal classical operation D_ε with $D_\varepsilon(p) = q_\varepsilon$. Analogous def. for quantum states (with ε).

Lemma: Let ρ, σ be quantum states on S with Hamiltonian H_S. If ρ, σ are blockdiagonal then

$$\underset{\text{quantumly}}{\rho \xrightarrow{\text{thermal}} \sigma} \iff \underset{\text{classically}}{\lambda(\rho) \xrightarrow{\text{thermal}} \lambda(\sigma)}$$

where $\lambda(\rho)$ are the energy eigenvalues of ρ, ordered in accordance with the ordering of the energy values.

Proof: maybe in homework.