Single-shot thermo, lecture 11

Recap last lecture:

Free energy as a transition rate:

$m_n :=$ largest integer with $A^{\otimes n} \xrightarrow{\varepsilon\text{-thermal}} B^{\otimes m_n}$, then

$$\lim_{n\to\infty} \frac{m_n}{n} = \frac{D(\rho_A \| \gamma)}{D(\rho_B \| \gamma)} = \frac{F_\beta(\rho_A) - F_\beta(\gamma)}{F_\beta(\rho_B) - F_\beta(\gamma)}$$

where $F_\beta(\rho) = \mathrm{tr}(\rho H) - S(\rho)/\beta$ (= "$U - TS$").

Reduction to classical case is not exactly possible (in contrast to resource theory of nonuniformity):

- not only eigenvalues of state ρ_S important, but also how eigenbasis of ρ_S "is rotated with respect to eigenbasis of H_S".
- Many equivalences of resource theory of nonunif. do not hold in the resource theory of athermality, cf.

$\rho \to \sigma$ by Gibbs-preserving map
$\not\Leftrightarrow$ $\rho \to \sigma$ by a thermal operation.

3.5. From the quantum to the classical case? Continued.

In light of these problems, let's stick to block-diagonal states, i.e. $[\rho_S, H_S] = 0$ $\rightarrow$ everything becomes classical.

Classical system: $A = (p_A, H_A)$

$p_A \in \mathbb{R}^n$ probability vector, $H_A = (E_1, \dots, E_n) \in \mathbb{R}^n$ energies

A permutation $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is <u>energy-preserving</u> if $\pi[H_A] = H_A$.

<u>Def</u>: A map $D : \mathbb{R}^{d_S} \to \mathbb{R}^{d_{S'}}$ is a <u>thermal classical operation</u> if there is a system $E = (\gamma_E, H_E)$ with $(\gamma_E)_i = \exp(-\beta (H_E)_i)/Z$ and an energy-preserving permutation π on SE with

$$D(p_S) = \left(\pi[p_S \otimes \gamma_E]\right)_{S'}.$$

$P = (p_S, H_S)$, $Q = (q_{S'}, H_{S'})$.

Write $P \xrightarrow{\text{thermal}} Q$ if for every $\varepsilon > 0$ there is q_ε with $\| q - q_\varepsilon \| < \varepsilon$ and a thermal operation D_ε with $D_\varepsilon(p) = q_\varepsilon$. Analogous def. hold for quantum states.

<u>Lemma</u>: Let ρ, σ be quantum states on S. [redacted] If ρ, σ are blockdiagonal then

$$\rho \underset{\text{quantumly}}{\xrightarrow{\text{thermal}}} \sigma \iff \lambda(\rho) \underset{\text{classically}}{\xrightarrow{\text{thermal}}} \lambda(\sigma)$$

where $\lambda(\rho)$ are the eigenvalues of ρ, ordered in accordance with the ordering of the energy values.

<u>Proof</u>: Maybe in homework.

<u>Theorem</u> (Janzing et al. 2000):

[redacted] Let $\mathbf{P} = (p_{\text{[redacted]}}, H_{\text{[redacted]}})$ and $\tilde{\mathbf{P}} = (\tilde{p}_{\text{[redacted]}}, \tilde{H}_{\text{[redacted]}})$ be classical systems of the same size, i.e. $p_{\text{[redacted]}}, \tilde{p}_{\text{[redacted]}} \in \mathbb{R}^n$.

Then

$$p \xrightarrow{\text{thermal}} \tilde{p}$$

if and only if there exists a stochastic matrix A such that $Ap = \tilde{p}$ and $Ag = \tilde{g}$,

where $g = \exp(-\beta H_{\text{[redacted]}})/Z$ and similarly $\tilde{g}$ are the Gibbs states.

Important theorem! This will be our main [redacted] tool in the following.

<u>Proof sketch</u>:

"$\Rightarrow$": In special case $H_{\text{[redacted]}} = \tilde{H}_{\text{[redacted]}}$, this means that thermal operations preserve the Gibbs state. Prove analogous to quantum case (cf. homework).

"$\Leftarrow$": [redacted]

Consider special case $n=2$ (i.e. bits).

$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ stochastic matrix, i.e. $a_{00} + a_{10} = a_{01} + a_{11} = 1$, $a_{ij} \geq 0$. Assume $Ap = \tilde{p}$ and $Ag = \tilde{g}$.

Consider the composite system with Hamiltonian

$$\underset{\text{input}}{H} + \underbrace{H + H + \dots + H}_{n} + \underbrace{\tilde{H} + \dots + \tilde{H}}_{n} + \underset{\text{output}}{\tilde{H}}$$

(in total $(2n+1)$ bits), and a thermal operation

$$D(\rho) = \left(\Pi\left[\rho \otimes g^{\otimes n} \otimes \tilde{g}^{\otimes(n+1)}\right]\right)_{2n+2}.$$

← want: this is $\approx \tilde{\rho}$.

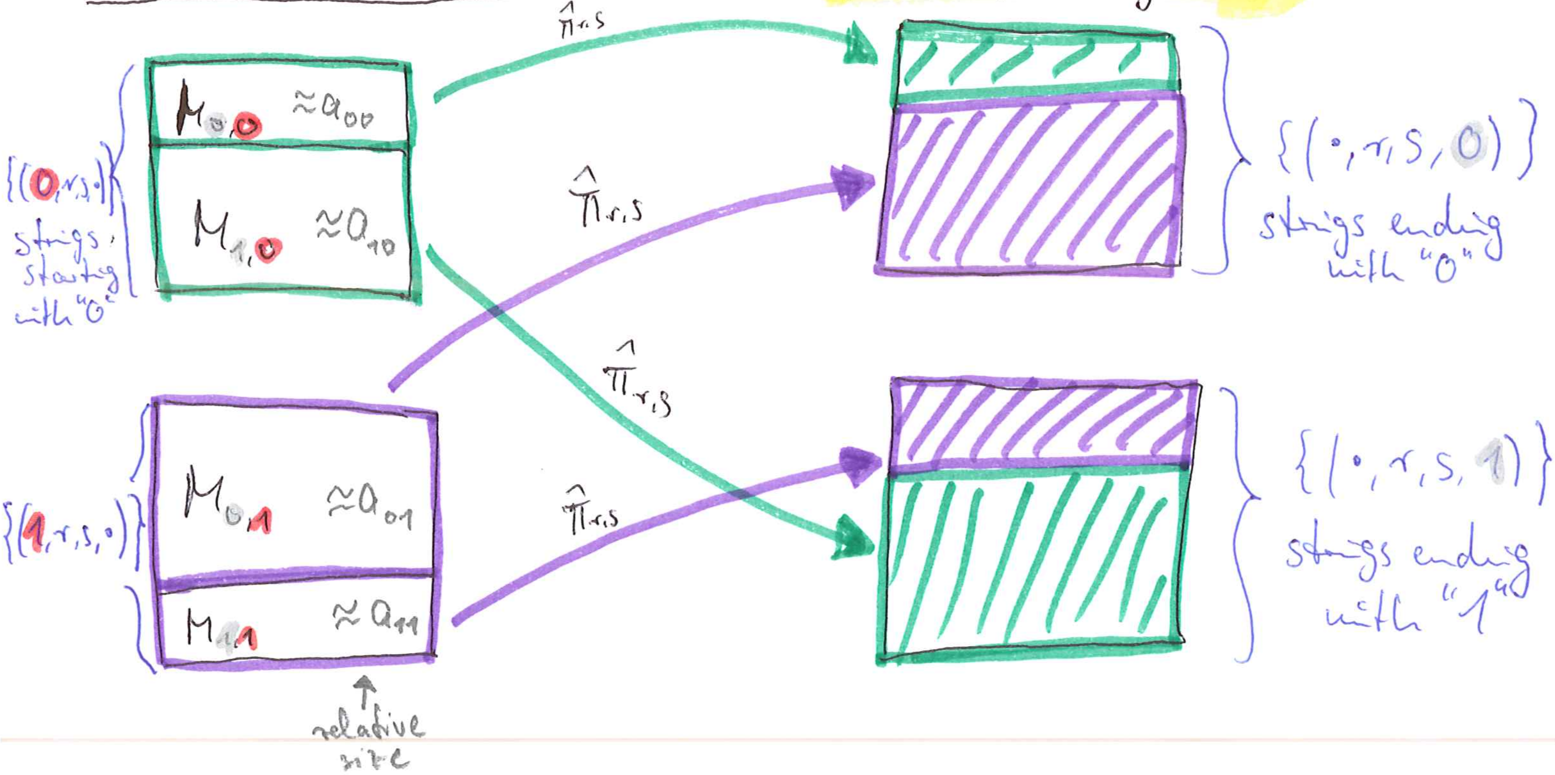
Construction of the energy-preserving permutation Π:

Two binary strings of length $(n+1)$ each are characterized by 4 numbers:

- first bit of the first word, denoted $j \in \{0,1\}$,
- number of ones in the first word, denoted $r \in [0, n+1]$
- " " " " " second ", denoted $s \in [0, n+1]$
- last bit of the last word, denoted $x \in \{0,1\}$.

→ 4-tuple (j, r, s, x).

Fix r, s. Then do [redacted] a permutation of the following kind:



- **Why would this give us what we want?**

If $n \to \infty$, we can restrict to <u>typical subset</u>.

$$T_n := \left\{ (j, r, s, x) \;\middle|\; \frac{r}{n} \approx g_1, \; \frac{s}{n} \approx \tilde{g}_1 \right\}$$

[redacted] approx. <u>uniform distribution</u> on T_n

$\Rightarrow$ prob. to be in some subset $\approx$ relative size of subset.

$\Rightarrow$ Prob (land in 0-box after permutation)

$= \text{Prob}(\text{start in } M_{00} \text{ or } M_{01})$

$\approx \text{Prob}(\text{start in green box}) \cdot a_{00}$

$\quad + \text{Prob}(\text{start in pink box}) \cdot a_{01}$

$\approx p_0 a_{00} + p_1 a_{01} = \sum_j a_{0j} p_j = (Ap)_0 = \tilde{p}_0$.

- **Check that we can indeed construct $\hat{\pi}_{r,s}$:**

Left boxes must fit (tightly) into right boxes,

Need
$$\left.\begin{aligned} \#\{(\cdot, r, s, 0)\} &\approx a_{00} \cdot \#\{(0, r, s, \cdot)\} \\ &\quad + a_{01} \cdot \#\{(1, r, s, \cdot)\}. \end{aligned}\right\} (*)$$

But $\#\{(\cdot, r, s, 0)\} \approx \tilde{g}_0 \cdot \#\{(\cdot, r, s, \cdot)\}$ (combinatorics for large n) (**)

[redacted] and $\#\{(j, r, s, \cdot)\} \approx g_j \cdot \#\{(\cdot, r, s, \cdot)\}$

Then $(*)$ follows from $\tilde{g}_0 = a_{00} g_0 + a_{01} g_1$,

i.e. $\tilde{g} = Ag$.

(**) Interlude: Consider n bits, **iid** $\sim p(1)=p$.

$$\frac{\#\{\text{strings with } s \text{ ones, last bit zero}\}}{\#\{\text{strings with } s \text{ ones, last bit arbitrary}\}} = \frac{\binom{n-1}{s}}{\binom{n}{s}}.$$

If $s \approx \alpha \cdot n$, then this tends to $(1-p)$ for $n \to \infty$.

3.6. d-majorization and [redacted] d-Lorenz curves

Def.: Let $p, p', d, d' \in \mathbb{R}^n$ be probability vectors. We say that (p,d) d-majorizes (p', d') if there is a channel (i.e. stochastic matrix) B with $Bp = p'$ and $Bd = d'$. Then we write: [redacted]

$$(p,d) \succ (p', d').$$

Special case $d = d'$, we also write $p \succ_d p'$.

$d = d' = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$: [redacted] standard majorization

Theorem: For probability vectors $p, p', d, d' \in \mathbb{R}^n$, the following conditions are equivalent:

(i) $(p, d) \succ (p', d')$

(ii) for every convex function g,

$$\sum_i d_i\, g\left(\frac{p_i}{d_i}\right) \geq \sum_i d_i'\, g\left(\frac{p_i'}{d_i'}\right),$$

(iii) the d-Lorenz curve of p is everywhere on or above the d'-Lorenz curve of p'.

(iv) $p \xrightarrow{\text{thermal}} p'$, if initial and final Hamiltonians are such that d and d' are the corresponding Gibbs states.

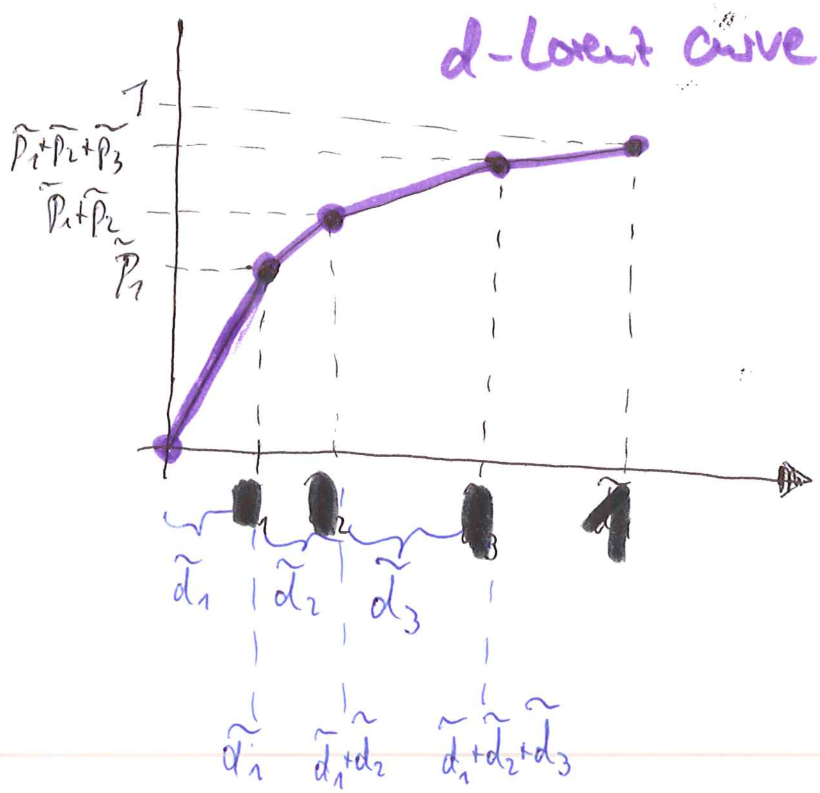
<u>Proof</u>: (i) $\Leftrightarrow$ (iv) is already proven.
Rest maybe in homework.

d-Lorenz curve (also "Gibbs-rescaled Lorenz curve"):

Consider $\left(\frac{p_i}{d_i}\right)_{i=1,\dots,n}$ and sort them in decreasing order:

$\tilde{p} = \pi(p)$, $\tilde{d} = \pi(d)$ (same permutation) such that

$$\frac{\tilde{p}_1}{\tilde{d}_1} \geq \frac{\tilde{p}_2}{\tilde{d}_2} \geq \frac{\tilde{p}_3}{\tilde{d}_3} \geq \dots \quad (***)$$



Curve is concave due to (***).

Consider condition (ii). For $\alpha > 1$, the function $g(x) := x^\alpha$ is convex, thus

$$\sum_i p_i^\alpha d_i^{1-\alpha} \geq \sum_i (p_i')^\alpha (d_i')^{1-\alpha}.$$

<u>Def</u>: (Relative Rényi entropy).
For $\alpha \in \mathbb{R} \setminus \{0,1\}$, set for prob. vectors $p, q \in \mathbb{R}^n$:

$$D_\alpha(p\|q) := \frac{\operatorname{sgn}\alpha}{\alpha - 1} \log \sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}.$$

Furthermore,

$$D_0(p\|q) := \lim_{\alpha \to 0^+} D_\alpha(p\|q) = -\log \sum_{i:\, p_i \neq 0} q_i$$

$$D_1(p\|q) = \lim_{\alpha \to 1} D_\alpha(p\|q) = \sum_{i=1}^n p_i(\log p_i - \log q_i)$$

standard relative entropy

$$D_\infty(p\|q) = \lim_{\alpha \to \infty} D_\alpha(p\|q) = \log \max_i \frac{p_i}{q_i}$$

$$D_{-\infty}(p\|q) = \lim_{\alpha \to -\infty} D_\alpha(p\|q) = D_\infty(q\|p).$$

cf. homework.

<u>Thm</u>: $(p, d) \succ (p', d') \Rightarrow$

$$D_\alpha(p\|d) \geq D_\alpha(p'\|d') \quad \forall \alpha.$$

$\Rightarrow$ Thermal operations <u>decrease</u> all relative-entropy-distances to the Gibbs state.