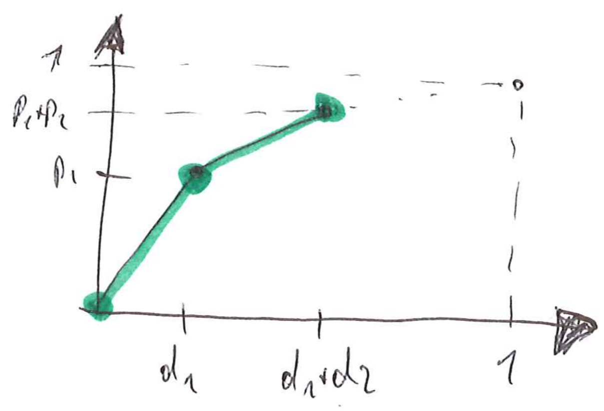
Single-shot thermo, lecture 12

- Sorry, exercise sheet 9 went online quite late
  → can give back until Thursday, 17:00 (to me or one of the secretaries in Philosophenweg 19).
- Exercise sheet 10 will be the last one.

Recap last lecture:

- For blockdiagonal states $[H, \rho]=0$: can reduce to the classical case.
- Equivalent are, classically:
  - $p \xrightarrow{\text{thermal}} p'$, if initial and final Hamiltonians are such that the Gibbs states are d and d',
  - $(p,d) \succ (p',d')$, i.e. $\exists$ stochastic B with $Bp=p'$ and $Bd=d'$.
  - Thermal Lorenz curve of p everywhere on or above that of p'.

Sort such that $\frac{p_1}{d_1} \geq \frac{p_2}{d_2} \geq \dots$ then



- for every convex fn. g:

$$\sum_i d_i\, g\left(\frac{p_i}{d_i}\right) \geq \sum_i d_i'\, g\left(\frac{p_i'}{d_i'}\right).$$

→ starting point for Rényi divergences

3.7. The Rényi divergence

Consider last condition for $g(x) := \pm x^\alpha$

<u>Def.</u> (Relative Rényi entropies)

If $p, q \in \mathbb{R}^n$ are probability vectors, and $\alpha \in \mathbb{R} \setminus \{0,1\}$,

$$D_\alpha(p \| q) := \frac{\operatorname{sgn}\alpha}{\alpha - 1} \log \sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}.$$

Furthermore,

$$D_0(p\|q) := \lim_{\alpha \to 0^+} D_\alpha(p\|q) = -\log \sum_{i:\, p_i \neq 0} q_i$$

cf. homework

$$D_1(p\|q) := \lim_{\alpha \to 1} D_\alpha(p\|q) = \sum_{i=1}^n p_i(\log p_i - \log q_i)$$

standard relative entropy $D(p\|q)$

$$D_\infty(p\|q) := \lim_{\alpha\to\infty} D_\alpha(p\|q) = \log \max_i \frac{p_i}{q_i}$$

$$D_{-\infty}(p\|q) := \lim_{\alpha \to -\infty} D_\alpha(p\|q) = D_\infty(q\|p).$$

<u>Thm.</u>: If $(p,d) > (p',d')$ then $D_\alpha(p\|d) \geq D_\alpha(p'\|d')$ $\forall \alpha$ [redacted]

<u>Notation</u>: $D_{min} = D_0$, $D = D_1$, $D_{max} = D_\infty$

$\Rightarrow D_{min} \leq D \leq D_{max}$.

Caution: $H_{min} = H_\infty$, $H = H_1$, $H_{max} = H_0$ (entropies)

$\Rightarrow H_{min} \leq H \leq H_{max}$. :(

3.8 Work of formation and the max-relative entropy

Suppose we want to <u>create</u> a system $P = (p[redacted], H')$.

prob. vector, resp. eigenvalues of block-diag. density matrix

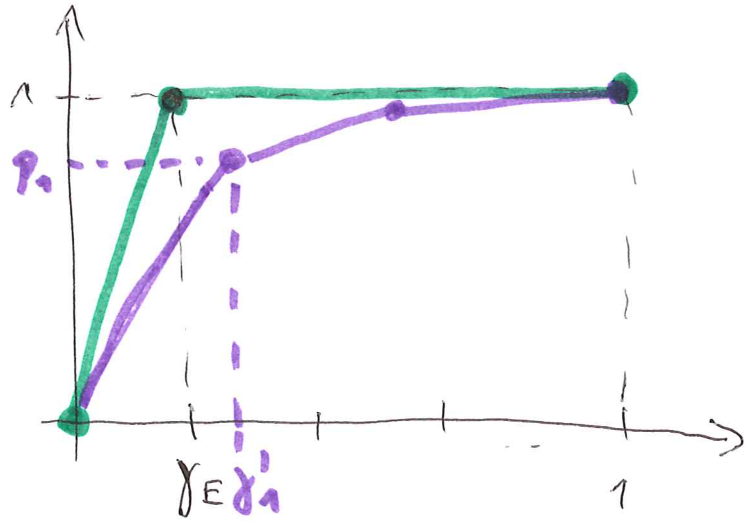
How much work does that cost?

→ start with energy eigenstate $|E\rangle$ of another Hamiltonian H.

Want $(|E[redacted]\rangle, H) \xrightarrow{\text{thermal}} (p, H')$.

What's the smallest possible E?

Thermo-Lorenz curve of $|E\rangle$ $(p_E = 1,$ all other $p_i = 0)$



curve of p

We only have to check the condition for the <u>initial</u> <u>slope</u>:

$$\frac{1}{\gamma_E} \geq \frac{p_1}{\gamma_1'} \text{ [redacted] } = \max_i \frac{p_i}{\gamma_i'} = e^{D_\infty(p \| \gamma')}$$

$$\gamma_E = e^{-\beta E} / Z$$

partition fn. of Hamiltonian H

$$\Rightarrow \boxed{E \geq \frac{1}{\beta} D_\infty(p \| \gamma') - \frac{1}{\beta} \log Z.} \quad (*)$$

If energy levels of H sufficiently dense, we can get (close to) equality.

Work of formation for thermal state:

$$\log Z = \log \sum_i e^{-\beta E_i} \geq \log e^{-\beta E_{min}} \geq -\beta E_{min}$$

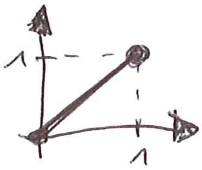
$$\Rightarrow -\frac{1}{\beta} \log Z \leq E_{min}$$

If $p = \gamma'$ then $D_\infty(p \| \gamma') = 0$ (Rényi divergences have this "distance measure" property) $\Rightarrow$

$$E \geq E_{min} \geq -\frac{1}{\beta} \log Z,$$

so (*) is automatically satisfied. **Thermal states are "free"!**

Lorentz curve of γ':



(*) is equivalent to

$$\boxed{D_\infty(\underbrace{|E\rangle\langle E|}_{\text{as prob. vector}} \| \gamma) \geq D_\infty(p \| \gamma')}$$

$$\Leftrightarrow F_\infty(|E\rangle\langle E|) - F_\infty(\gamma) \geq F_\infty(p) - F_\infty(\gamma')$$

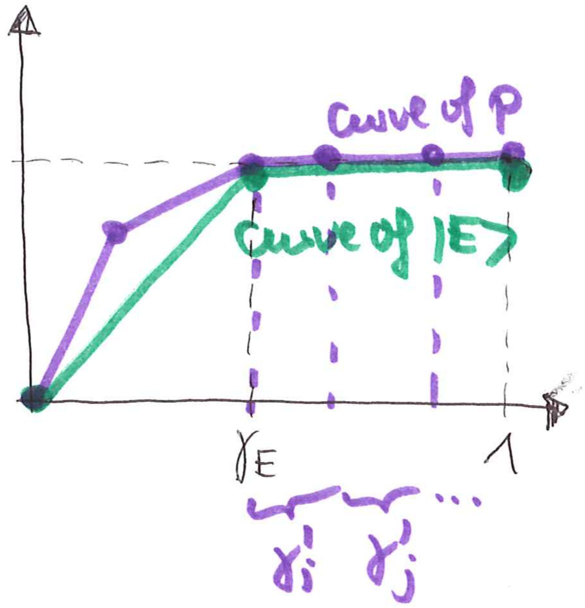
where $F_\infty(p) := \frac{1}{\beta} D_\infty(p \| \gamma') - \frac{1}{\beta} \log Z'$

(if the system is $P = (p, H')$ with γ' Gibbs state)

"max-free energy" $F_\infty =: F_{max}$

3.9. Extractable work and the min-relative entropy

Suppose we are given a system $P = (p, H')$ and want to create a system $(|E\rangle, H)$.
What's the largest possible E we can get?



Need "flat tail" $\hat{=}$ zero entries of P.

green tail length $\leq$ pink tail length

$$\Rightarrow 1 - \gamma_E = 1 - \frac{e^{-\beta E}}{Z} \leq \sum_{i:\, p_i = 0} \frac{e^{-\beta E_i'}}{Z'} = 1 - \sum_{i:\, p_i \neq 0} \gamma_i'$$

$$= 1 - e^{-D_0(p \| \gamma')}$$

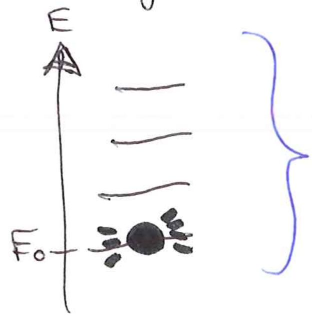
$$\Rightarrow \boxed{E \leq \frac{1}{\beta} D_0(p \| \gamma') - \frac{1}{\beta} \log Z} \qquad (**)$$

If we choose H suitably, we can achieve equality.

If $\rho = \gamma'$: $D_0(\rho \| \gamma') = 0 \Rightarrow$

$$E \leq -\frac{1}{\beta} \log Z \leq E_{\min}$$

$\Rightarrow$ Can either get the ground state, or no pure state at all.



does not contain useful work.

(**) is equivalent to

[redacted]

$$\boxed{D_0(\rho \| \gamma') \geq D_0(|E\rangle\langle E|, \gamma)}$$

$$\Leftrightarrow F_0(\rho) - F_0(\gamma') \geq F_0(|E\rangle\langle E|) - F_0(\gamma)$$

where $F_0(\rho) := \frac{1}{\beta} D_0(\rho \| \gamma') - \frac{1}{\beta} \log Z'$

is the "min-free energy" $F_0 =: F_{\min}$.

In general $\underset{D_{\min}}{\overset{\shortparallel}{D_0}} \ll \underset{D_{\max}}{\overset{\shortparallel}{D_\infty}}$

fundamental asymmetry

$$\boxed{\begin{aligned} \Rightarrow E_{\text{extractable}} &\leq \frac{1}{\beta} D_0(\rho \| \gamma') - \frac{1}{\beta} \log Z \\ &\ll \frac{1}{\beta} D_\infty(\rho \| \gamma') - \frac{1}{\beta} \log Z \leq E_{\text{formation}}. \end{aligned}}$$

3.10. Allowing finite error probability: Smoothing

Introduce "smoothed Rényi divergences":

$$D_0^\varepsilon(p\|q) := \max_{p' : D(p,p') \le \varepsilon} D_0(p'\|q),$$

$$D_\infty^\varepsilon(p\|q) := \min_{p' : D(p,p') \le \varepsilon} D_\infty(p'\|q).$$

Approximate work of formation: Want

$$(|E\rangle, H) \xrightarrow{\text{thermal}} (p', H') \text{ with } D(p,p') \le \varepsilon.$$

Minimal work needed is therefore

$$W^\varepsilon_{form} = \min_{p' : D(p,p') \le \varepsilon} \left(\frac{1}{\beta} D_\infty(p'\|\gamma') - \frac{1}{\beta} \log Z \right)$$

$$\Rightarrow \boxed{W^\varepsilon_{form} = [redacted] \frac{1}{\beta} D_\infty^\varepsilon(p\|\gamma') - \frac{1}{\beta} \log Z}$$

Approximate extractable work:

Want $(p, H') \xrightarrow{\text{thermal}} (q, H)$ with $D(q, |E\rangle\langle E|) \le \varepsilon$.

Choose p' with $D_0^\varepsilon(p\|\gamma') = D_0(p'\|\gamma')$, $D(p,p') \le \varepsilon$.

$\Rightarrow$ Can obtain $(|E\rangle\langle E|, H)$ by thermal op. ϕ from p', where

$$E \le \frac{1}{\beta} D_0(p'\|\gamma') - \frac{1}{\beta} \log Z = \frac{1}{\beta} D_0^\varepsilon(p\|\gamma') - \frac{1}{\beta} \log Z.$$

$$\Rightarrow D(\phi(p), |E\rangle\langle E|) = D(\phi(p), \phi(p')) \le D(p,p') \le \varepsilon.$$

$$\Rightarrow \boxed{W^{\varepsilon}_{extr} \geq \frac{1}{\beta} D^{\varepsilon}_{0}(p \| \gamma') - \frac{1}{\beta} \log Z.}$$

Exact expression is a bit harder to get; maybe homework.