ÖAW AUSTRIAN ACADEMY OF SCIENCES

IQOQI – INSTITUTE FOR QUANTUM OPTICS AND QUANTUM INFORMATION VIENNA

Probabilistic Theories and Generalized Contextuality: When is a Physical System Genuinely Quantum?

Markus P. Müller

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PERIMETER INSTITUTE

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Quantum Information and Foundations of Physics

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Overview on these lectures

Guiding question: How can you prove rigorously, from experimental data alone, that your physical system is nonclassical?

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Lectures: partially slides, partially on the blackboard.

QM as a theory of probability

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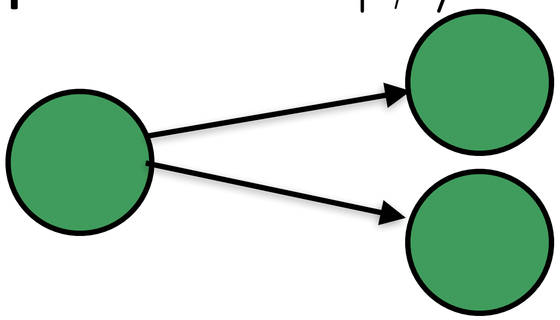
There are no physical maps that implement $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$:



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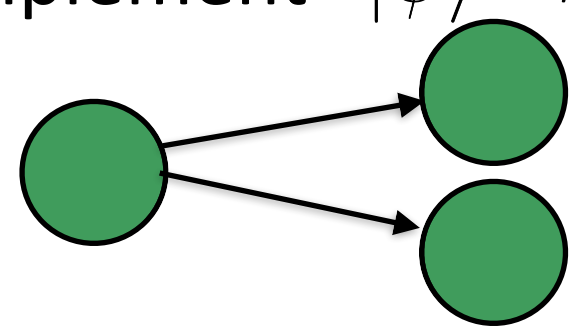

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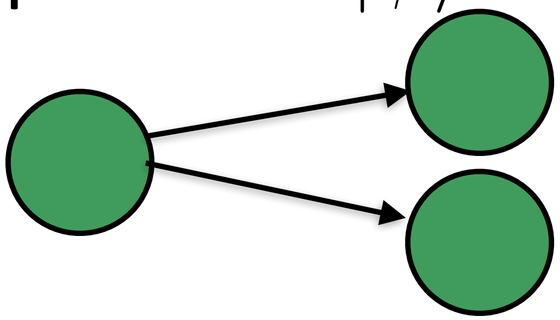

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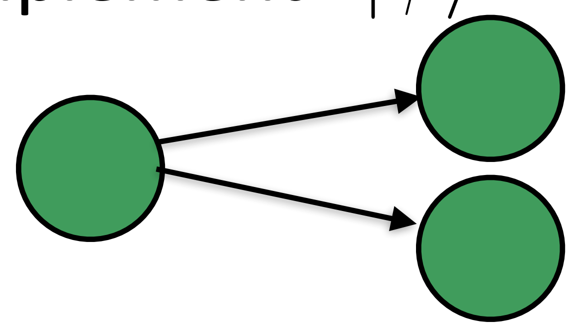

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All probabilistic theories have these (and other) properties in common.

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Long prehistory of unsuccessful (but insightful) attempts:

- Via **quantum logic:** Birkhoff & von Neumann (1936), Piron…
- As a **probabilistic** theory: Günther Ludwig (1970s/80s) (pre-QIT)
- Lucien Hardy 2001: “QM From Five Reasonable Axioms”
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Motivation for this lecture: GPTs are a powerful technical tool.

More general than quantum?

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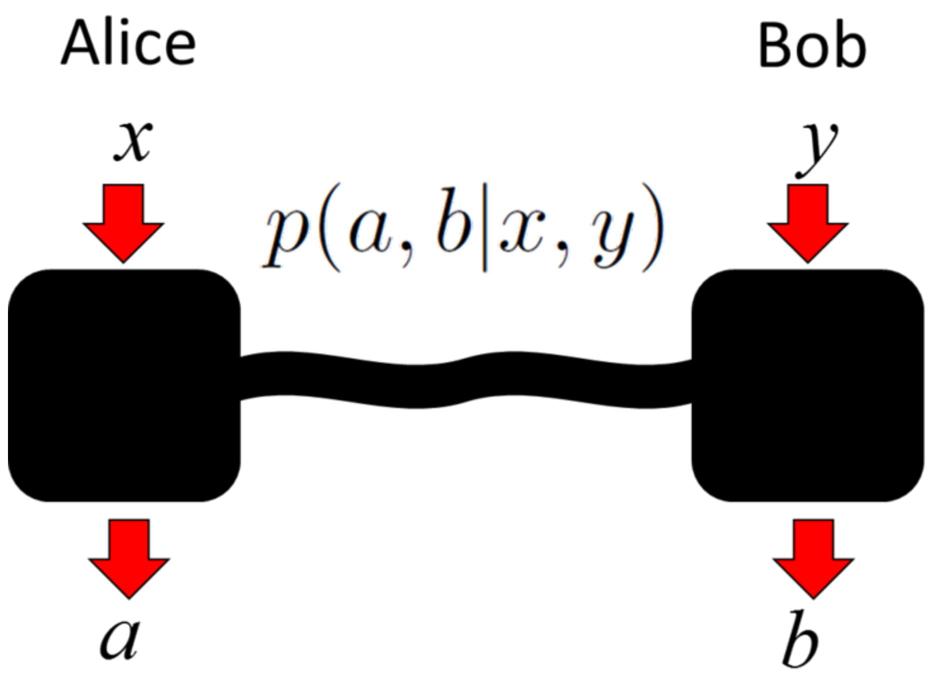

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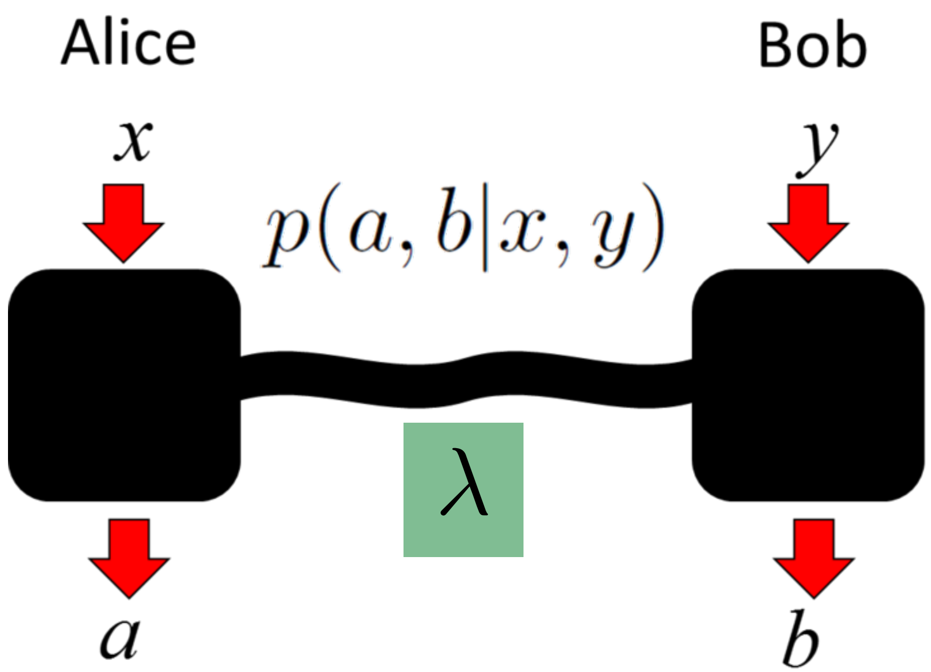

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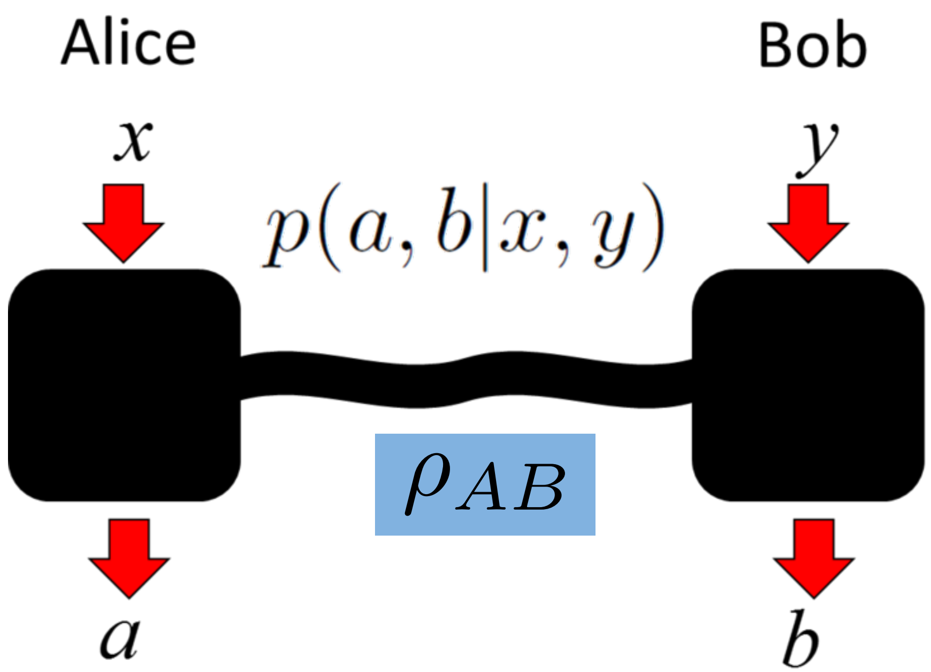

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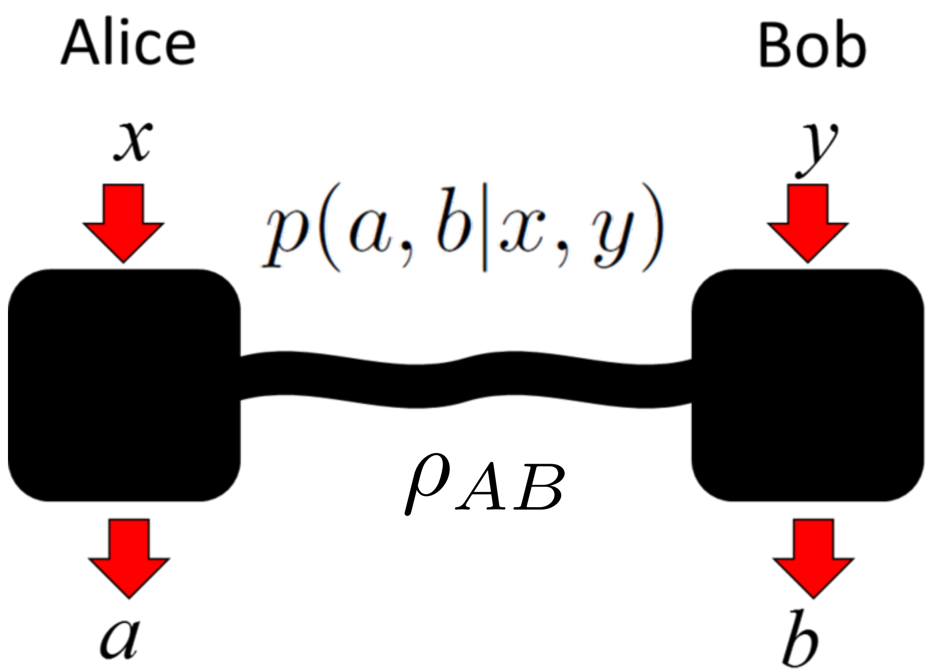

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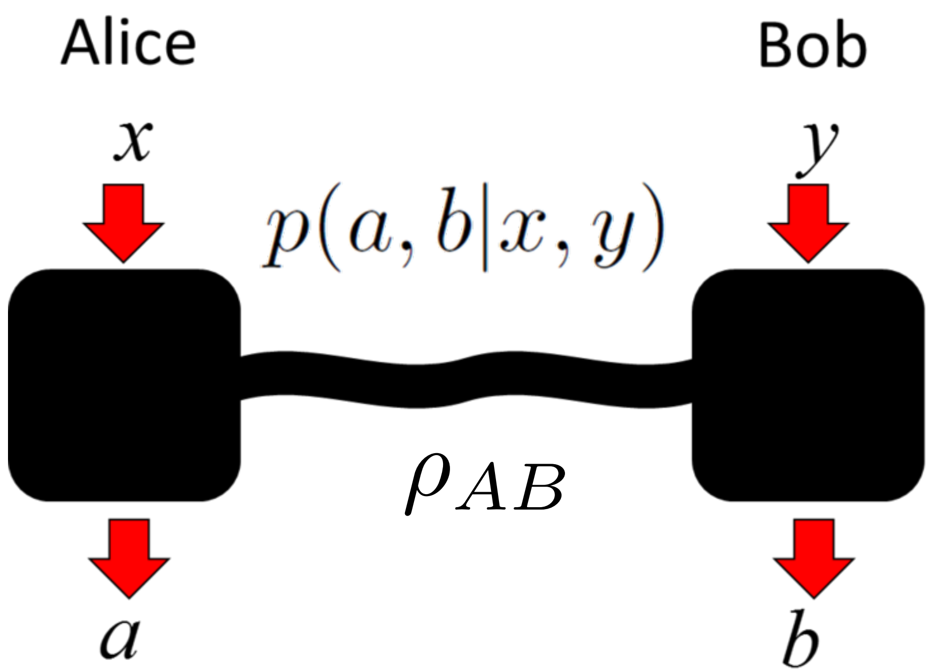

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Quantum theory admits more general *P*'s due to the **violation of Bell inequalities**.

The Bell-CHSH inequality

Classical probability distributions satisfy Bell inequality:

$$\mathrm{CHSH} := |C_{00} + C_{01} + C_{10} - C_{11}| \leq 2 \quad \text{where} \quad C_{xy} := \mathbb{E}(a \cdot b | x, y).$$

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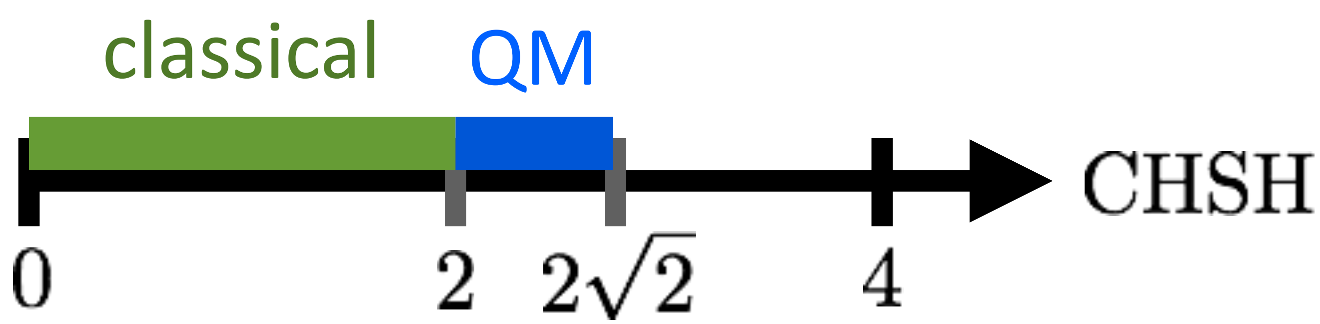

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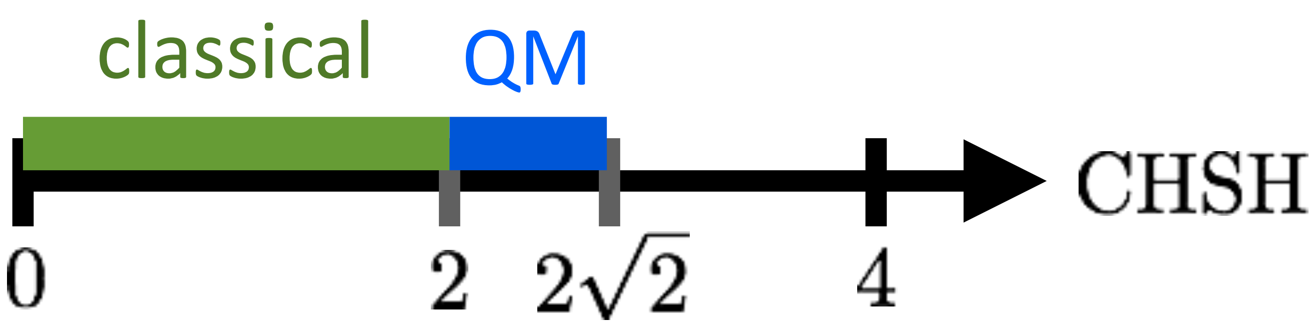

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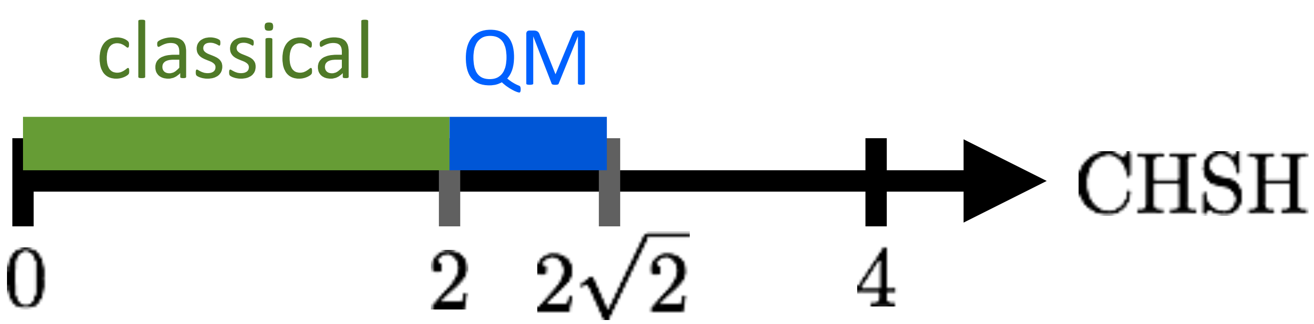

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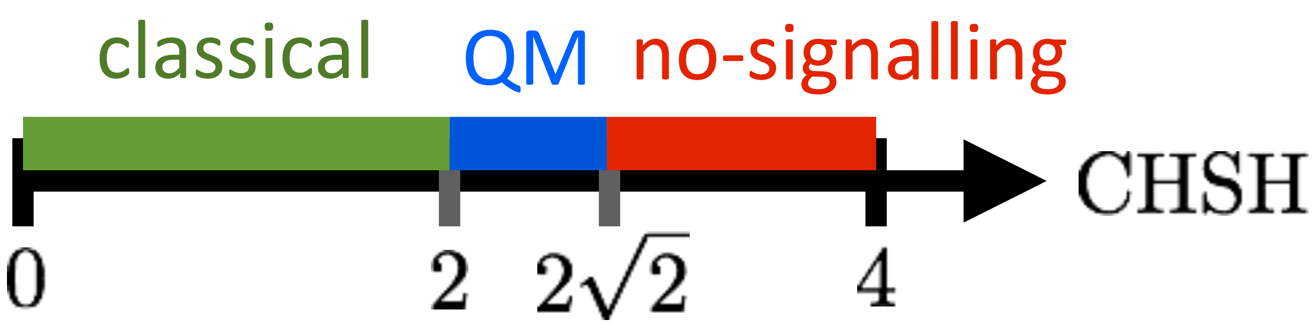

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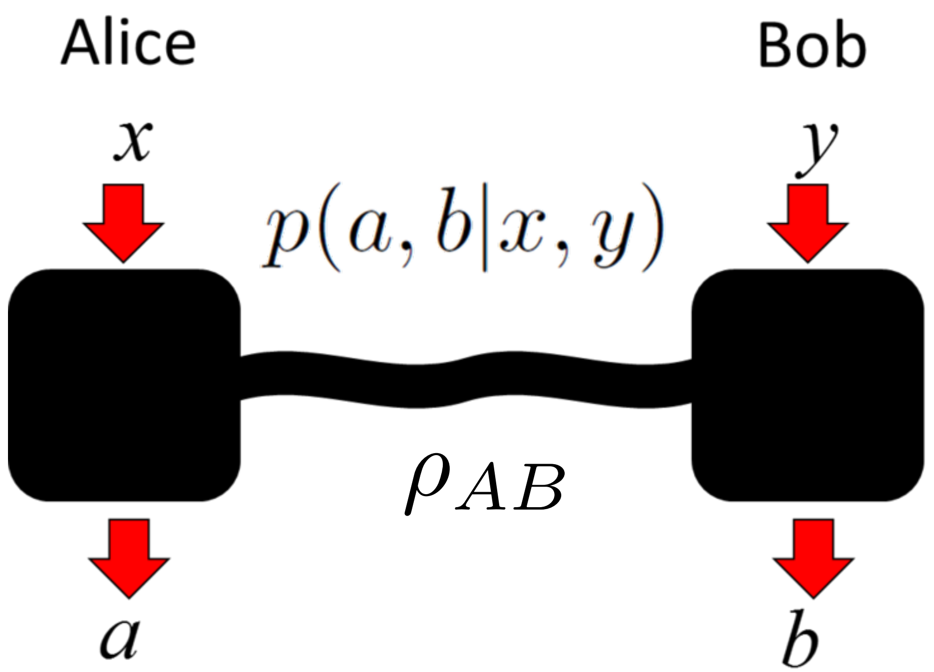

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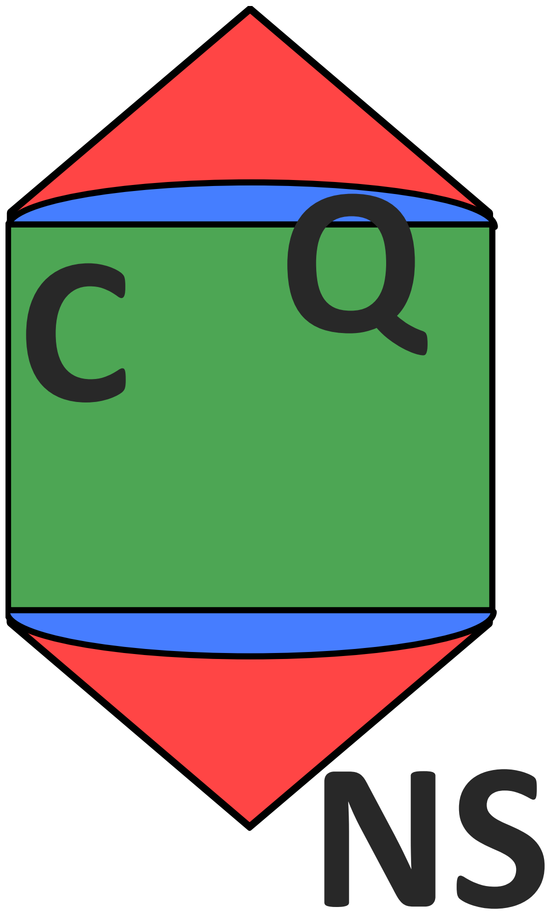

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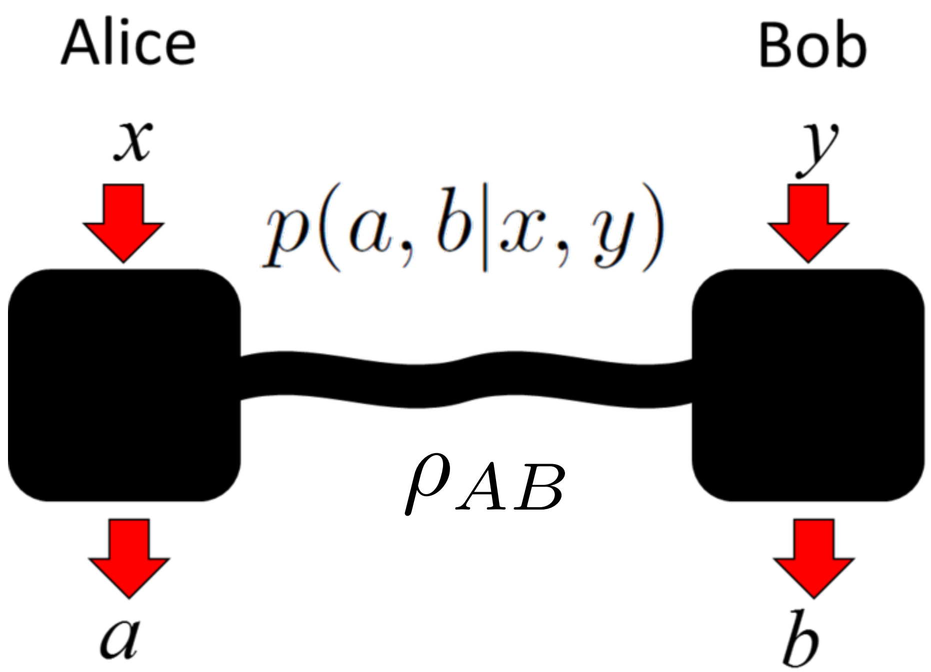

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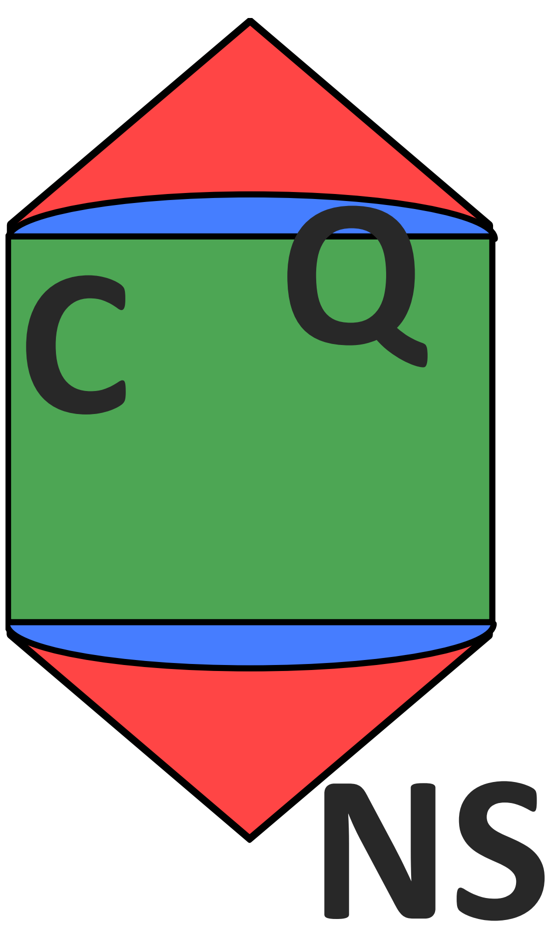

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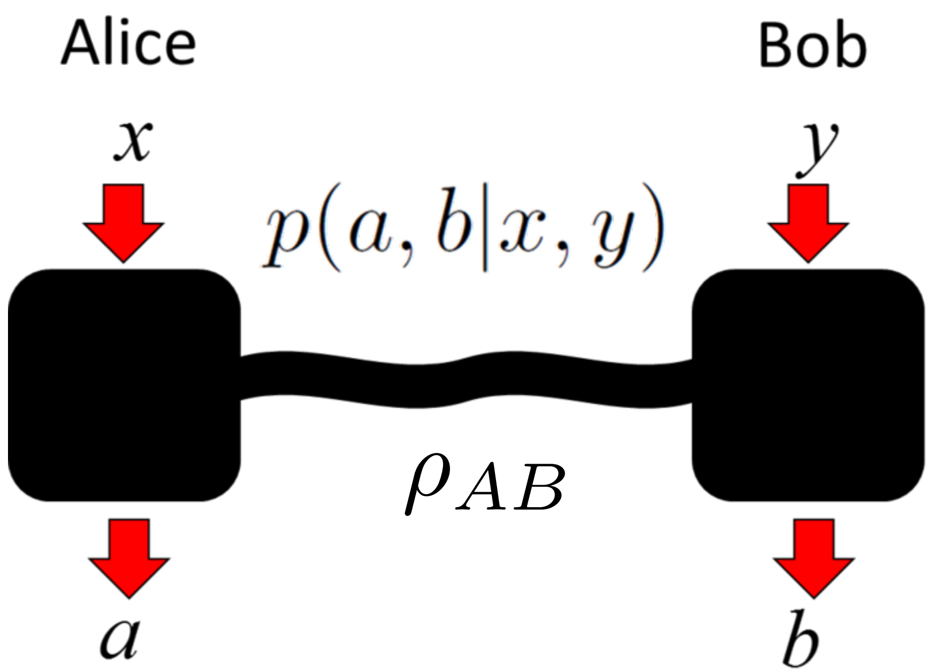

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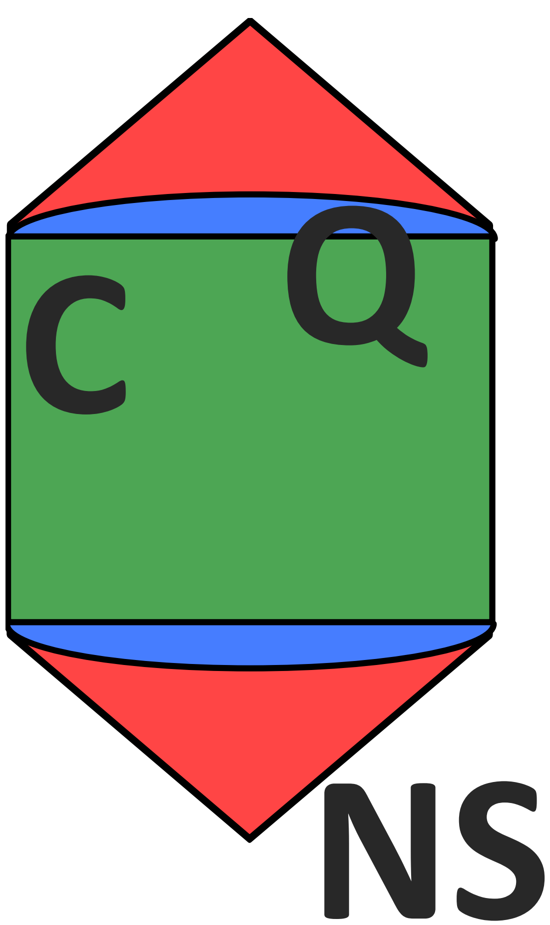

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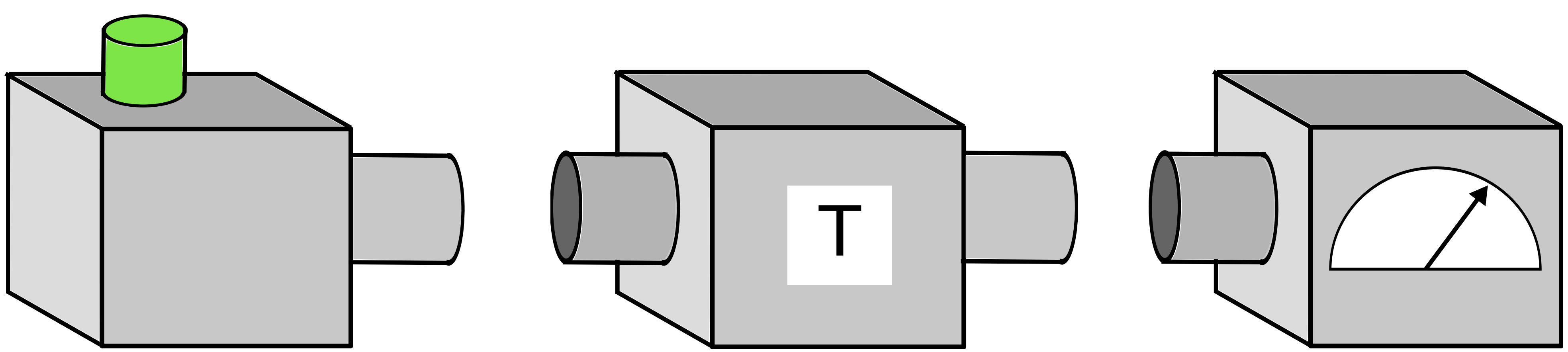
3 examples of a "**generalized probabilistic theory**".

Generalized probabilistic theories



Preparation transformation measurement

Generalized probabilistic theories

Example: classical coin toss.

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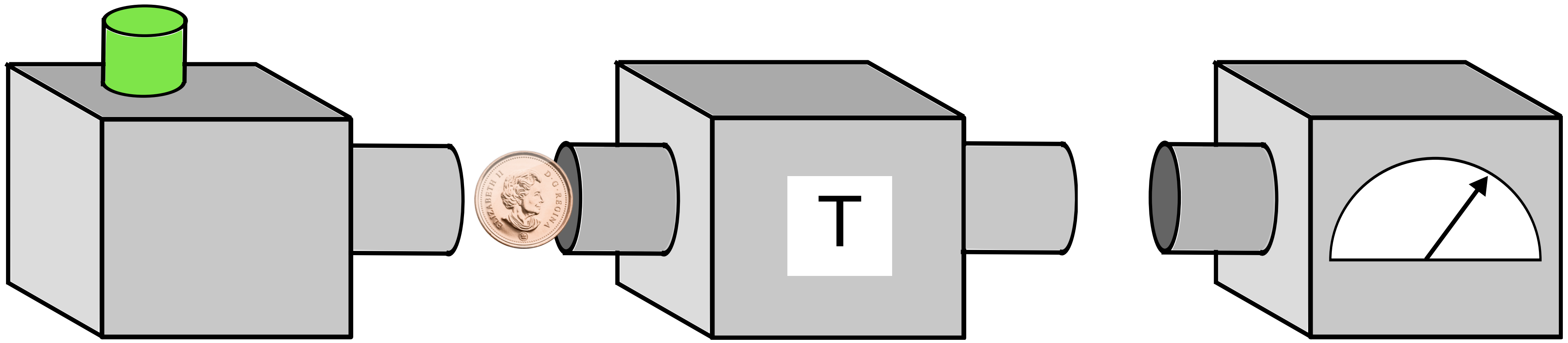

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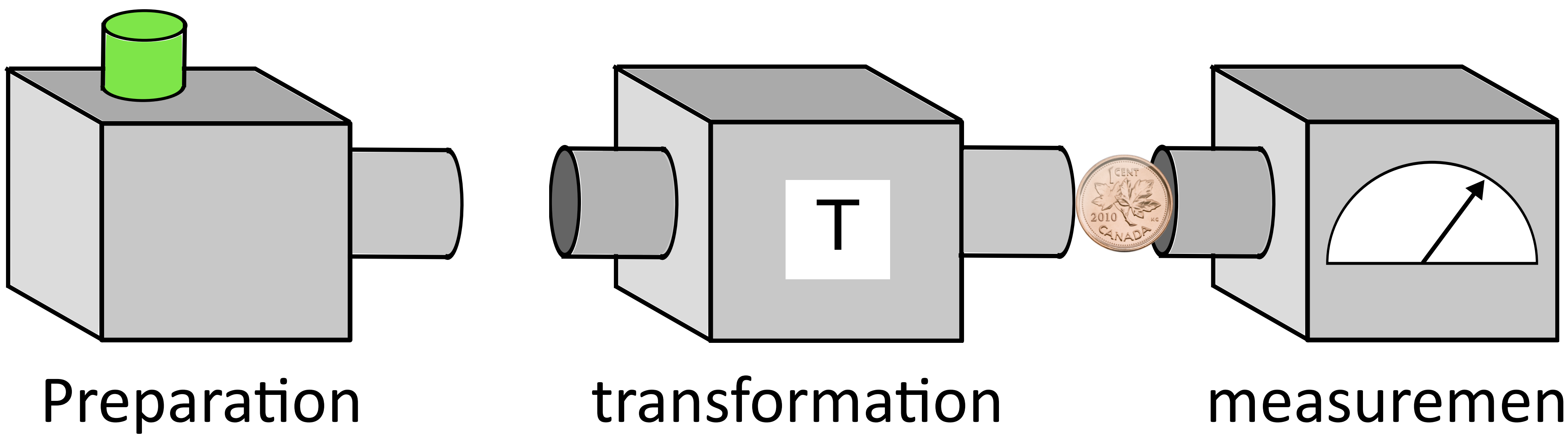

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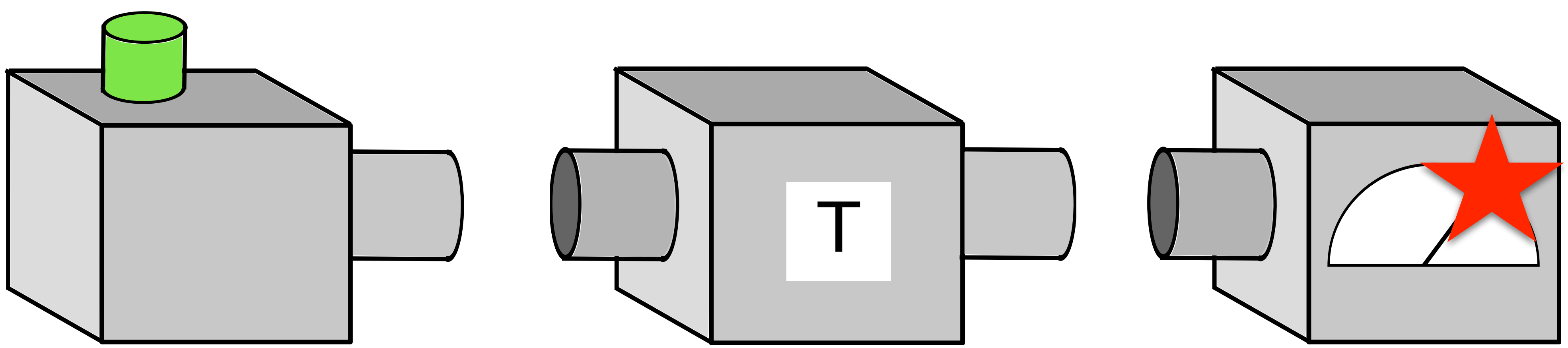
- The measurement outcome is "heads" or "tails".



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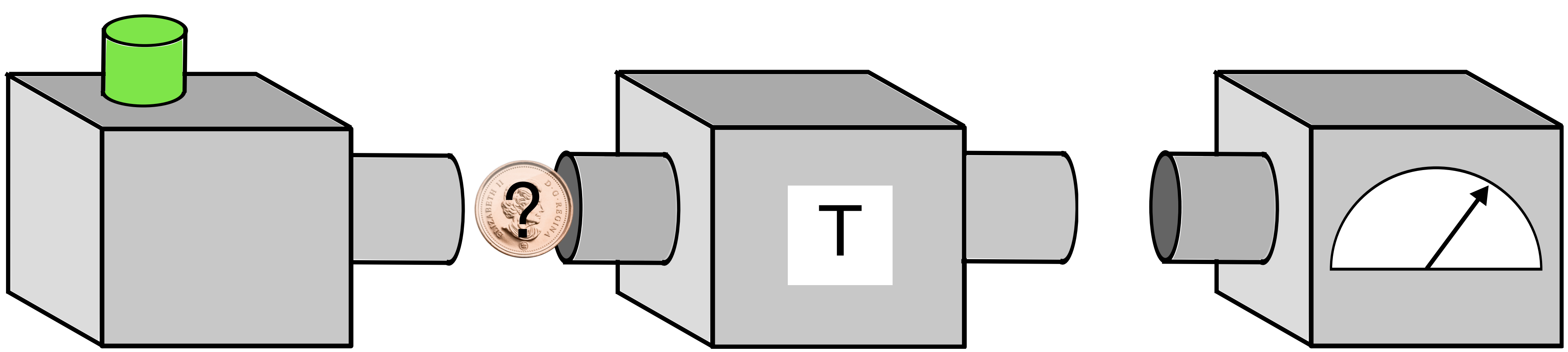

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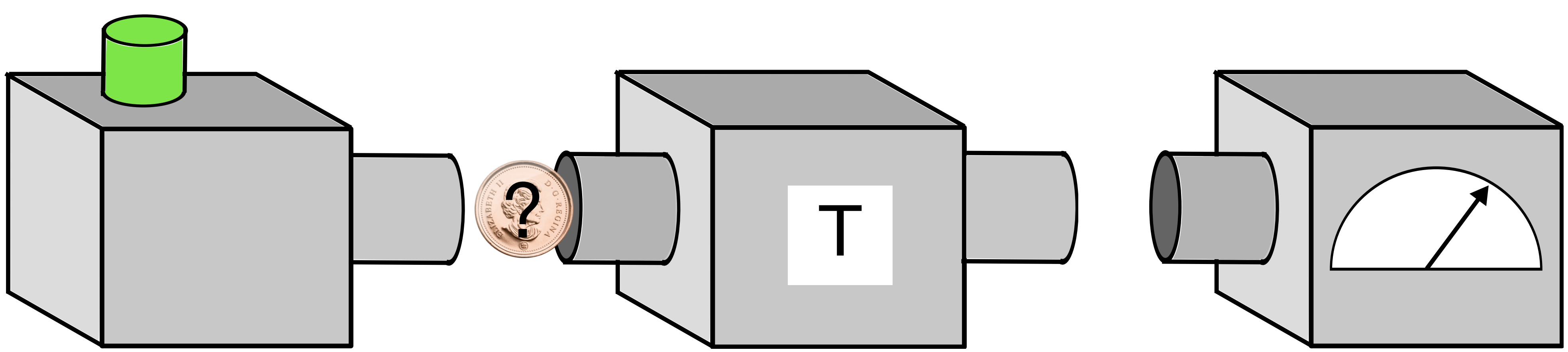

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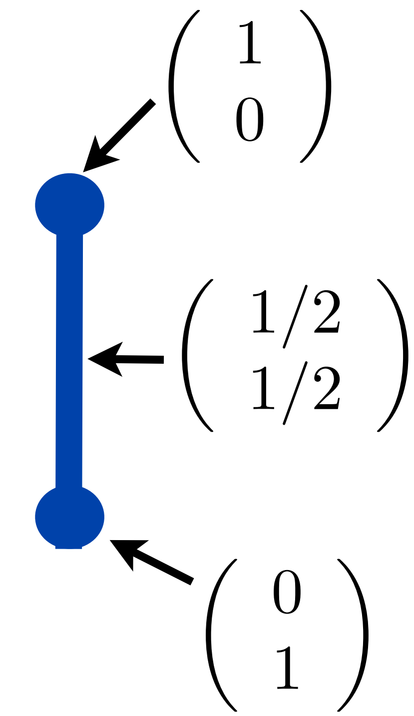

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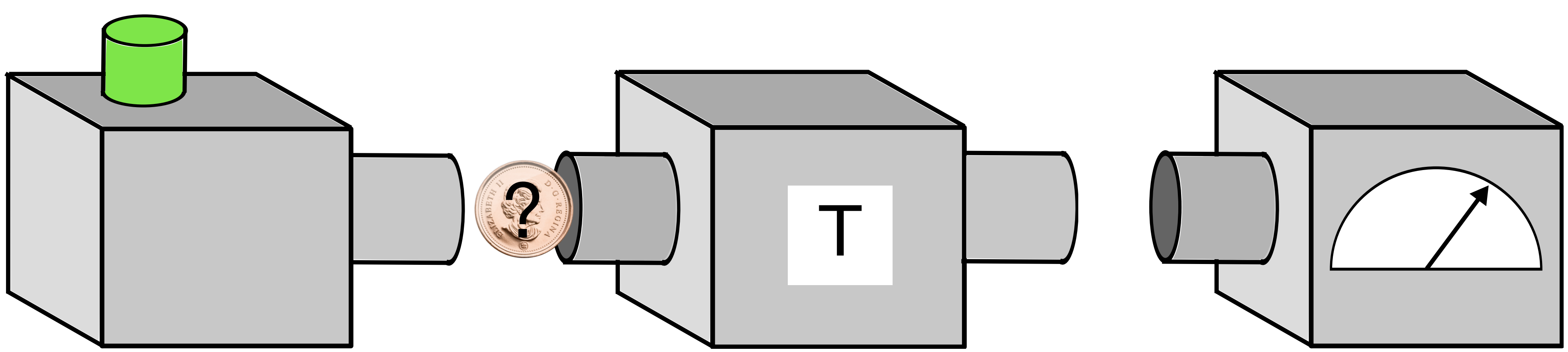

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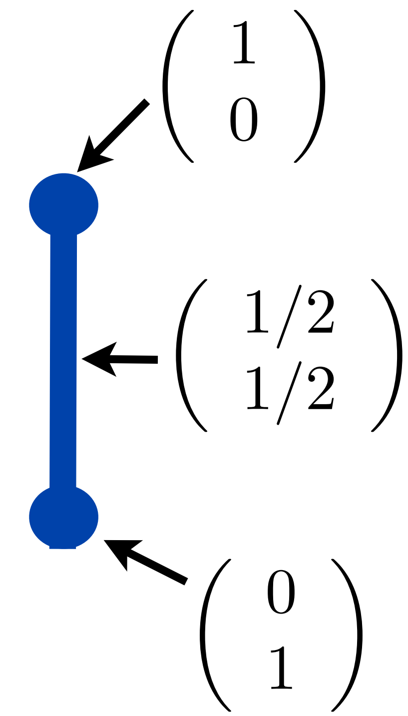

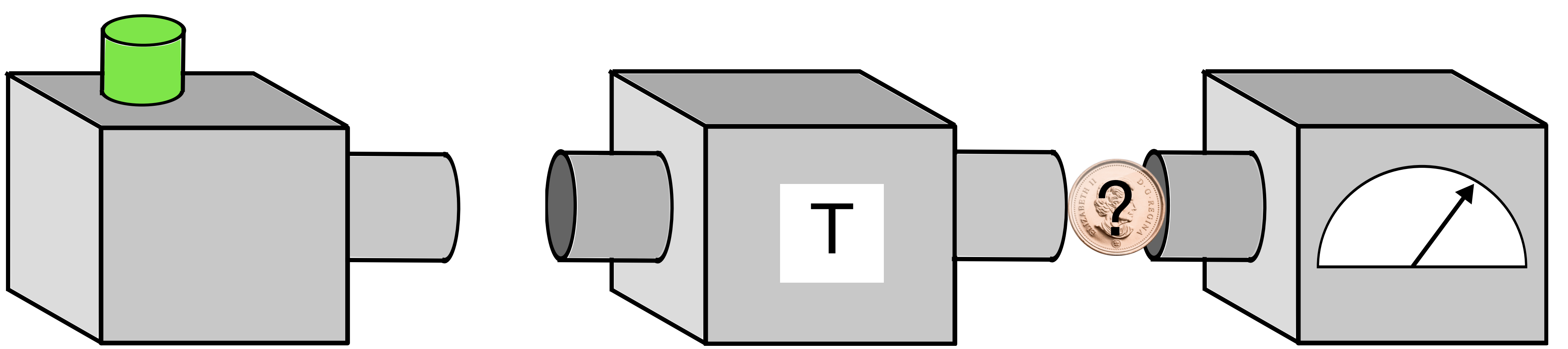

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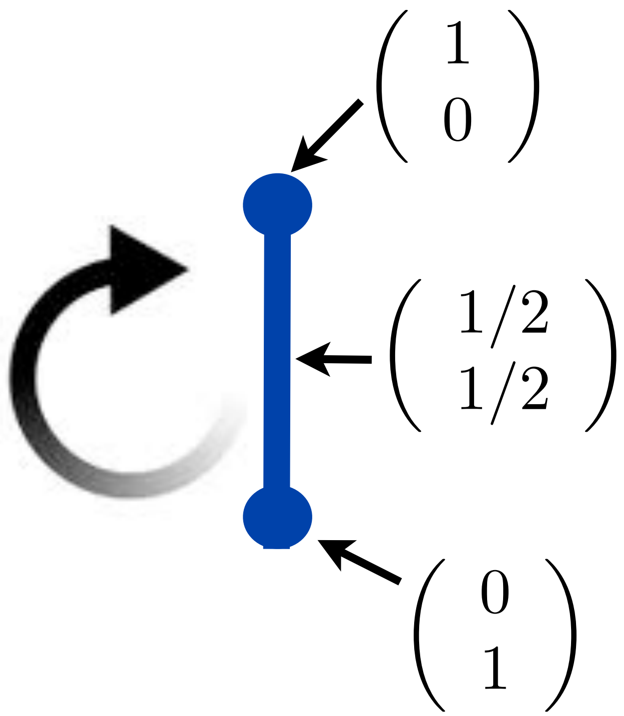

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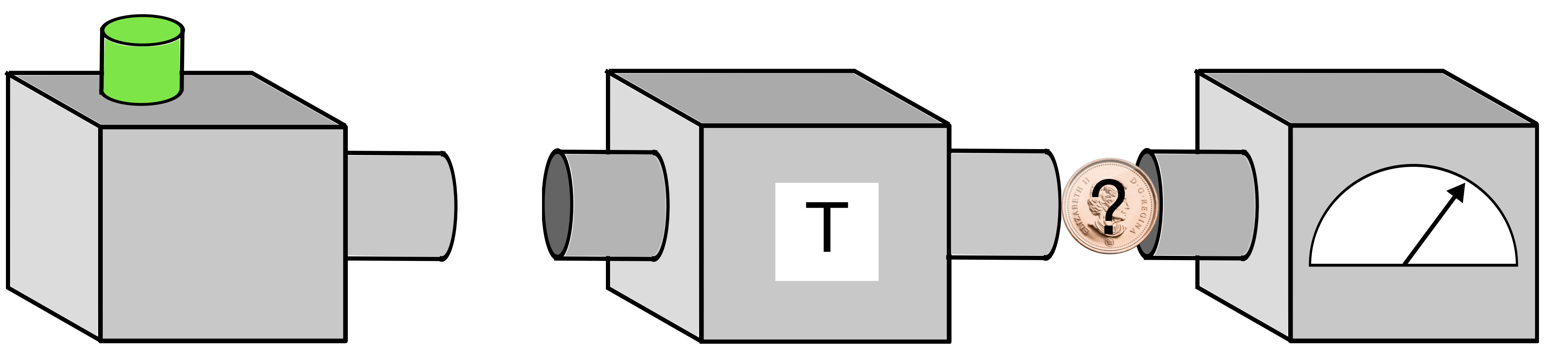

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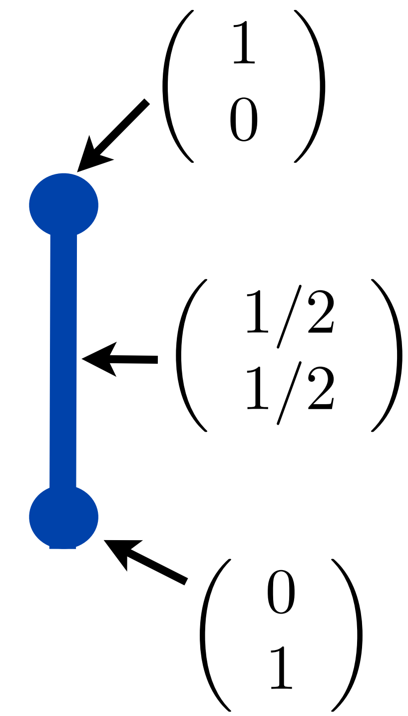

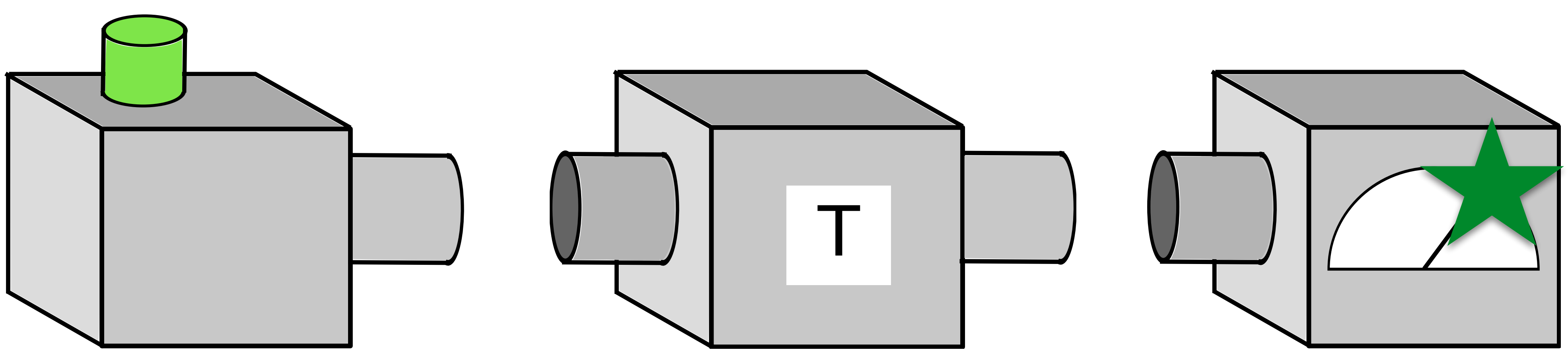

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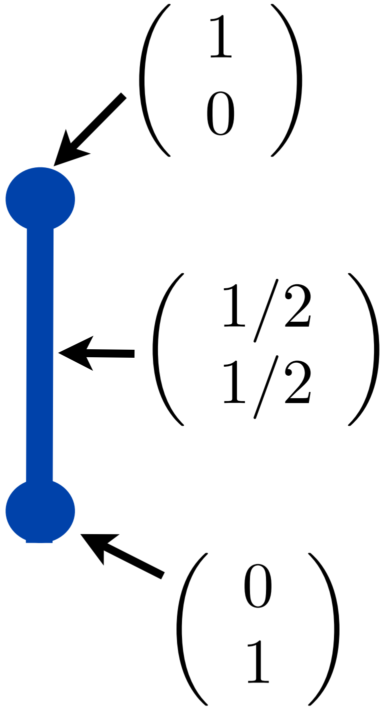
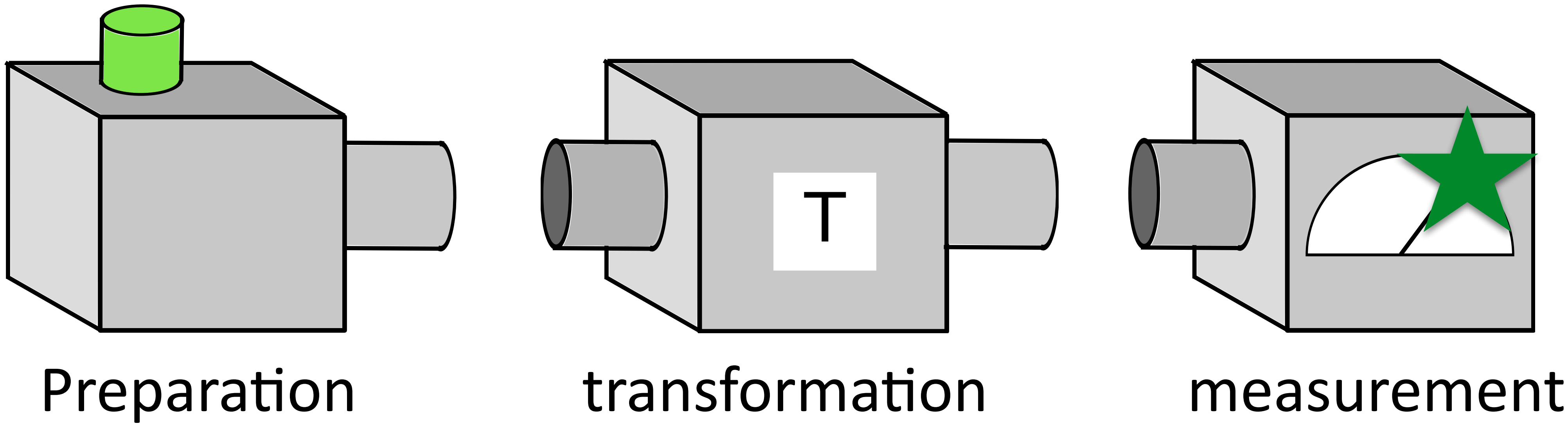
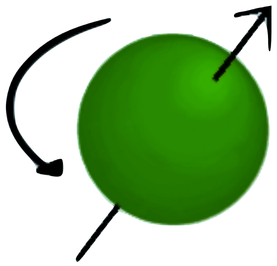
$$\mathrm{Prob}(\mathrm{heads}|\omega) = p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ 1-p \end{pmatrix} = e \cdot \omega .$$

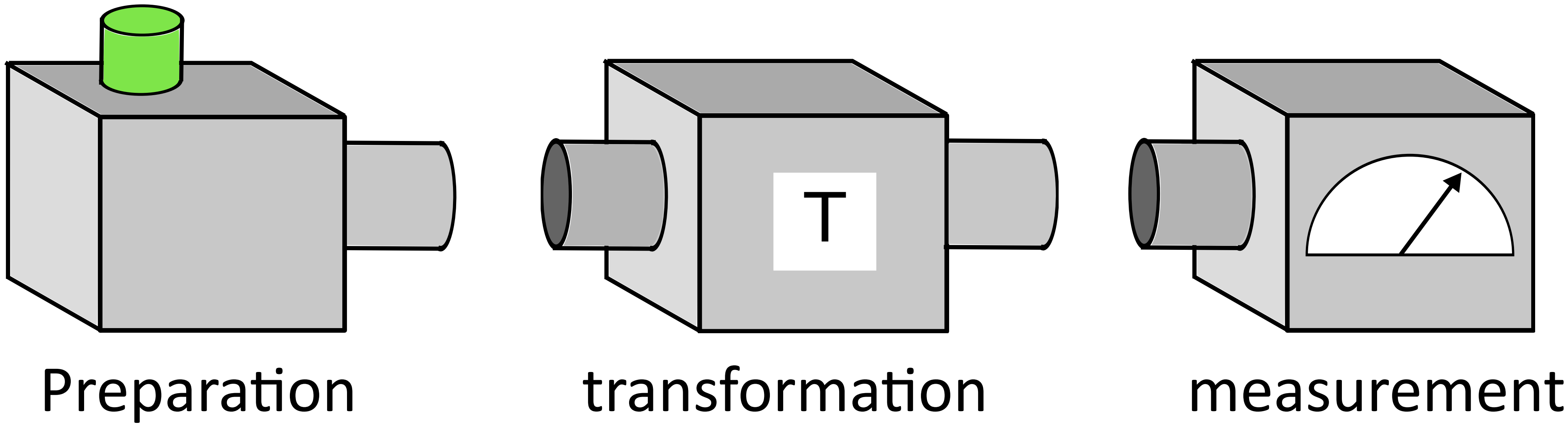




Generalized probabilistic theories

Example: quantum bit.





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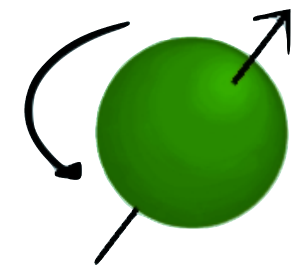
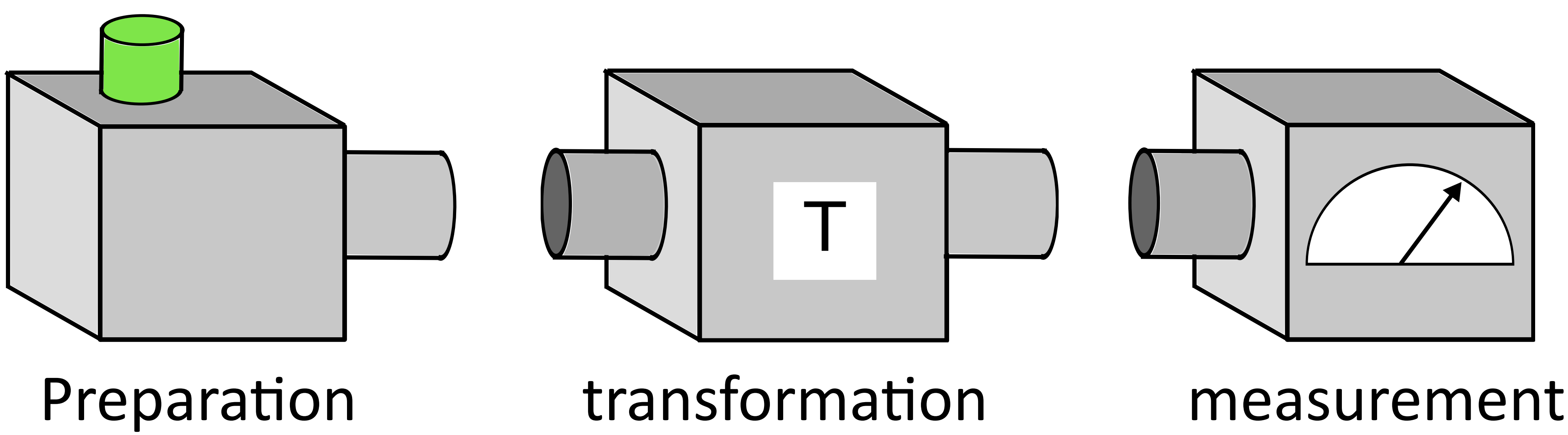


- The preparation device prepares a qubit in some quantum state $\rho = |\psi\rangle\langle\psi|,$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

More generally: ρ is (mixed-state-)2x2 density matrix.



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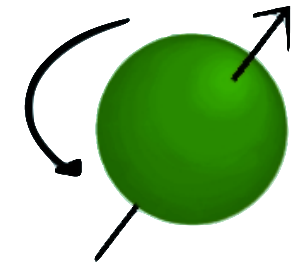

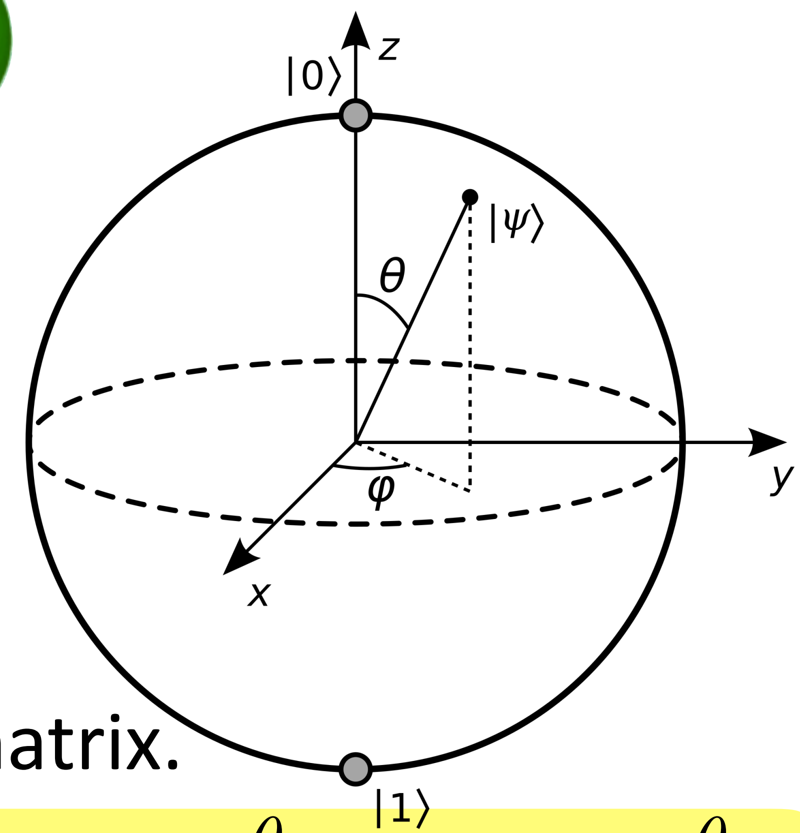

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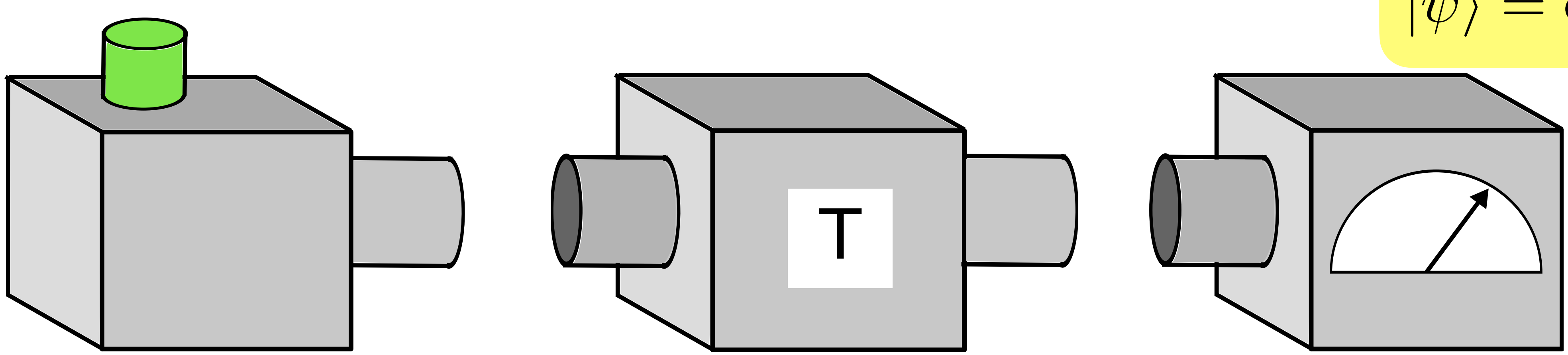

Preparation transformation measurement

Generalized probabilistic theories

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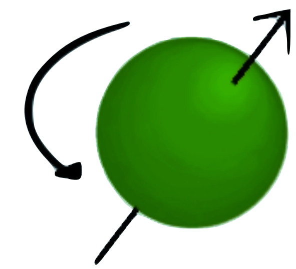

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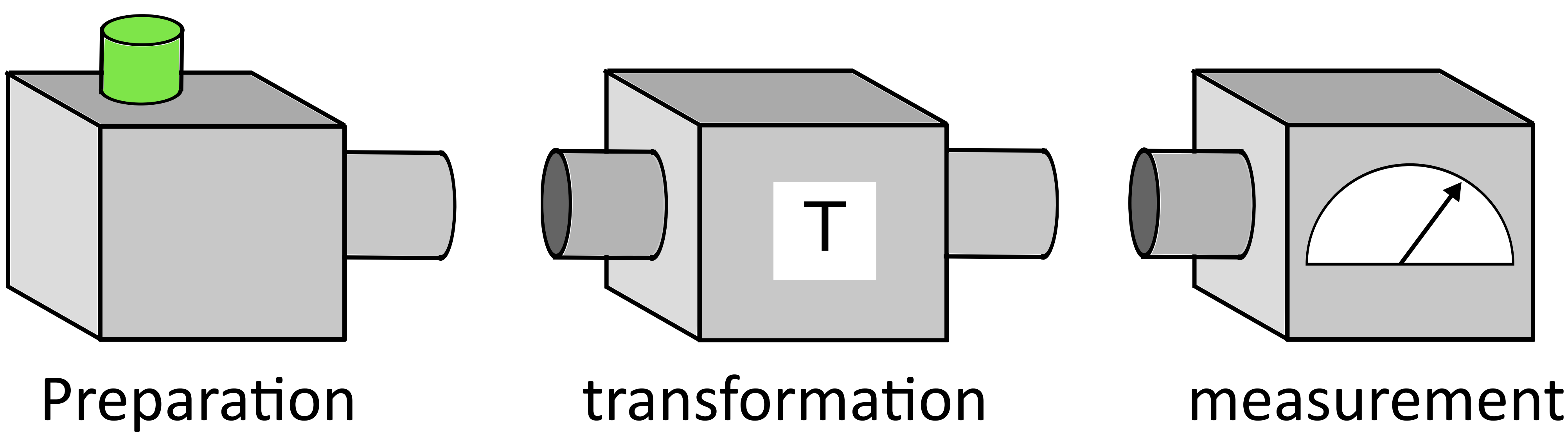

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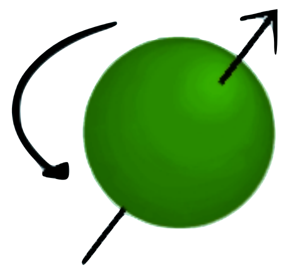

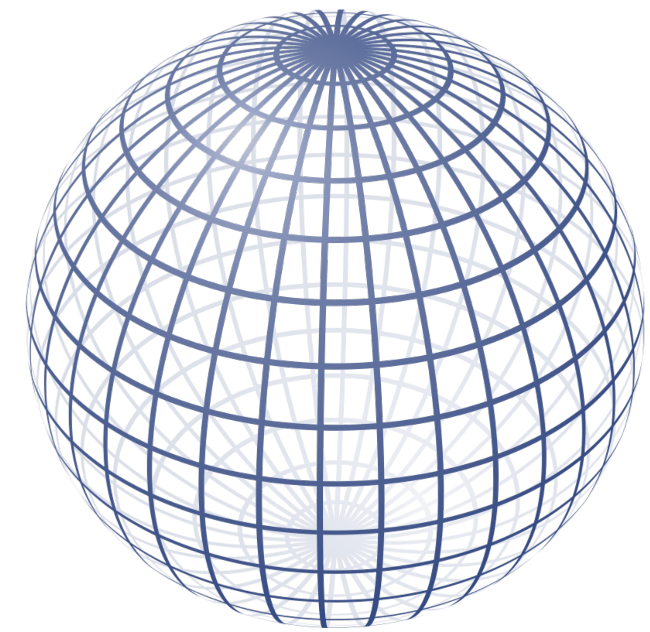

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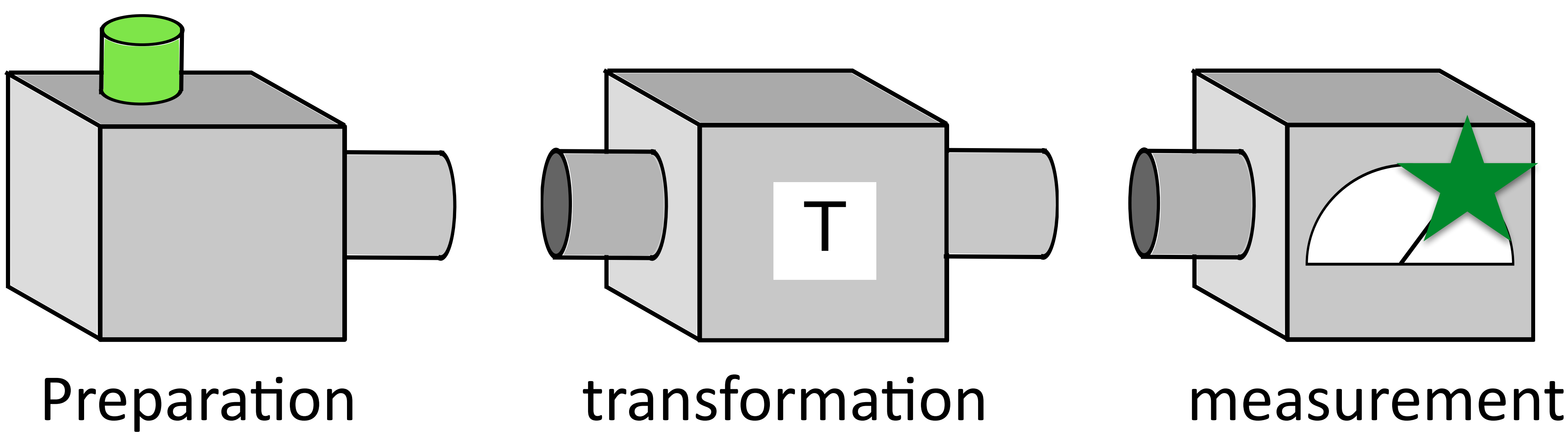
- Measurement in arbitrary spin direction *d*:

$$\mathrm{Prob}(\uparrow|\rho) = \mathrm{tr}(P_d \rho).$$



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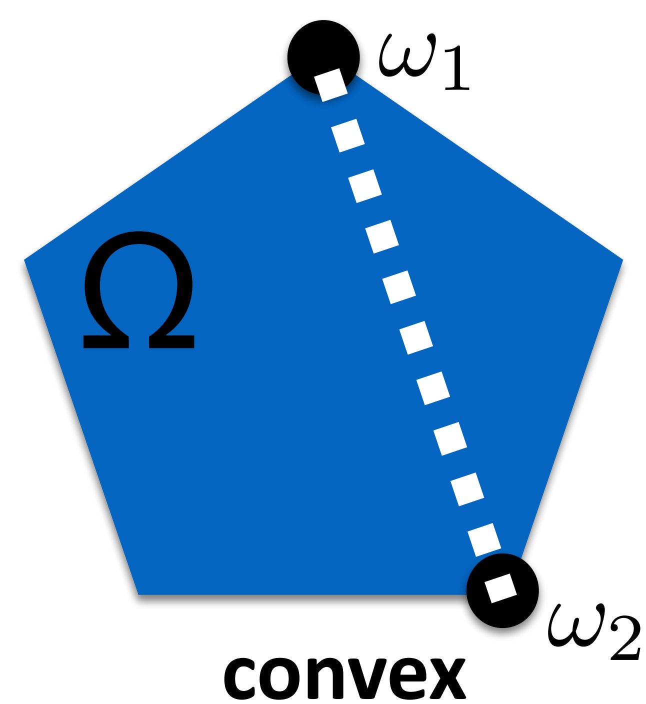

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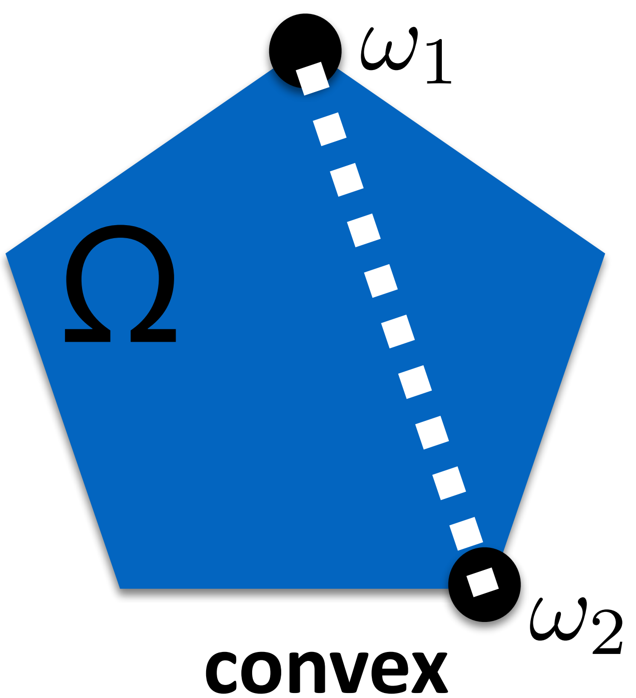

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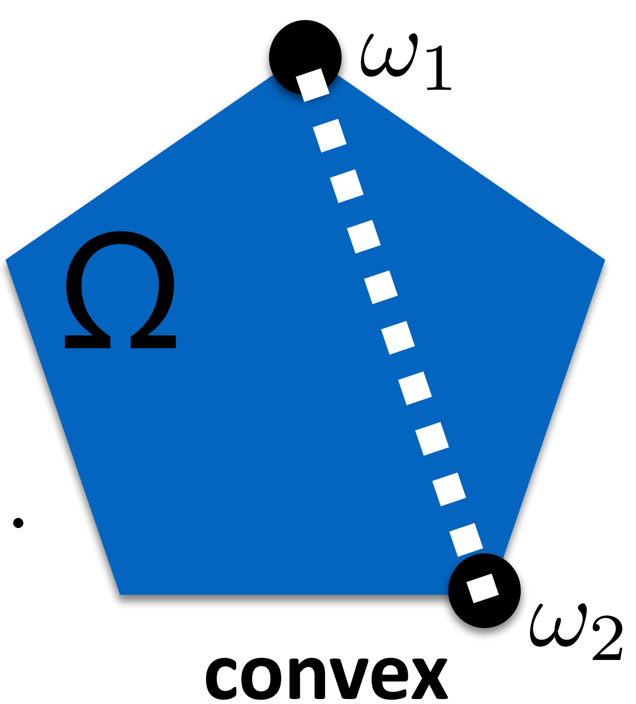

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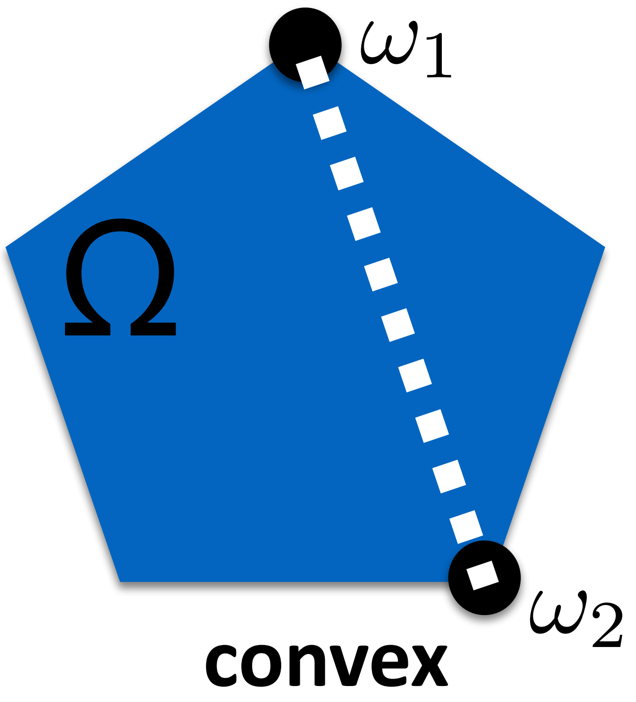

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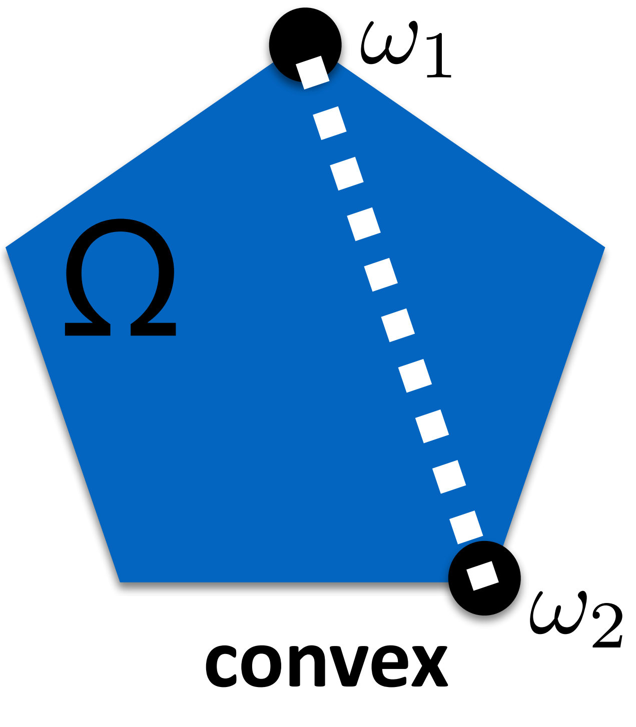

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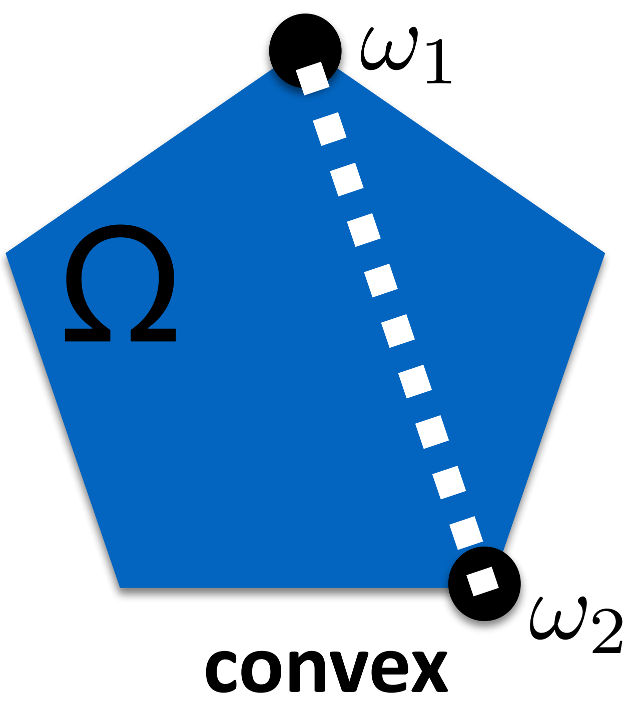

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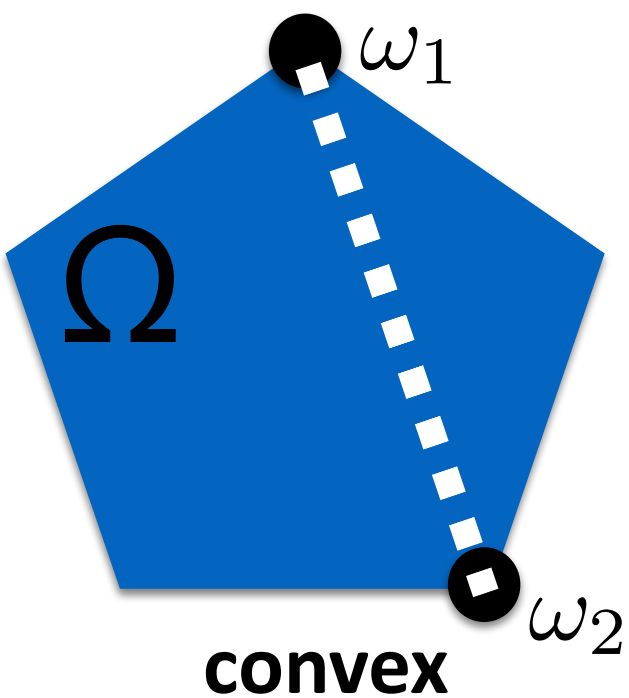
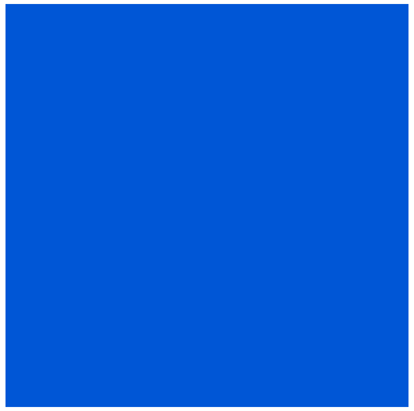


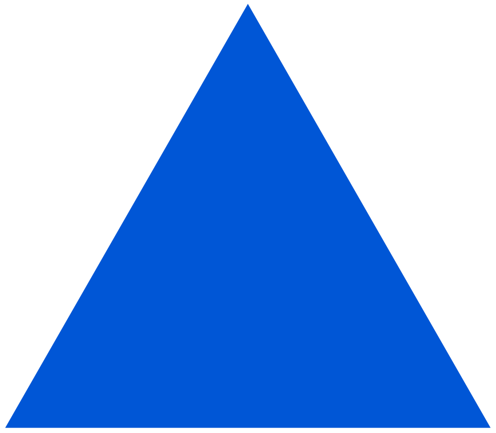
Generalized probabilistic theories

classical
bit

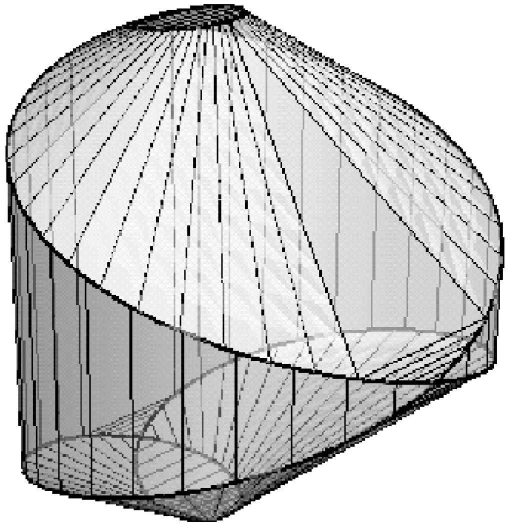
quantum
bit



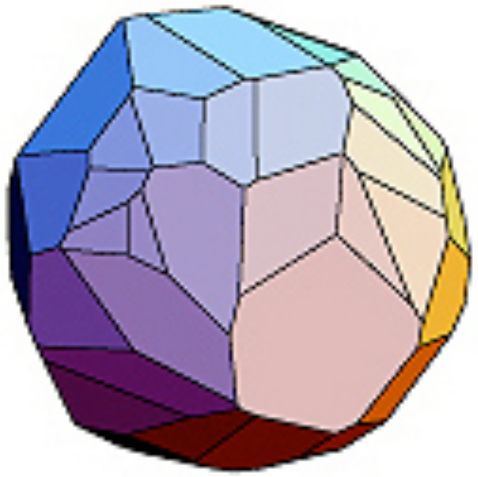
"gbit"



Classical trit
(3-level-system)



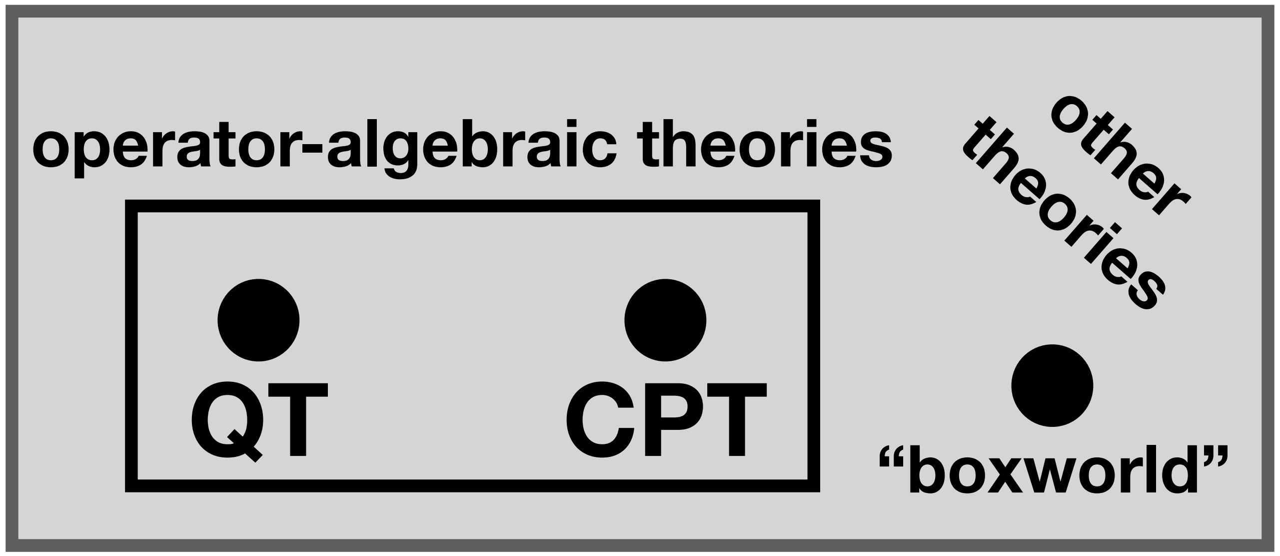
Quantum trit:
8D and complicated!



Arbitrary convex
state space

Generalized probabilistic theories

There is a large landscape of generalized probabilistic theories or GPT systems:



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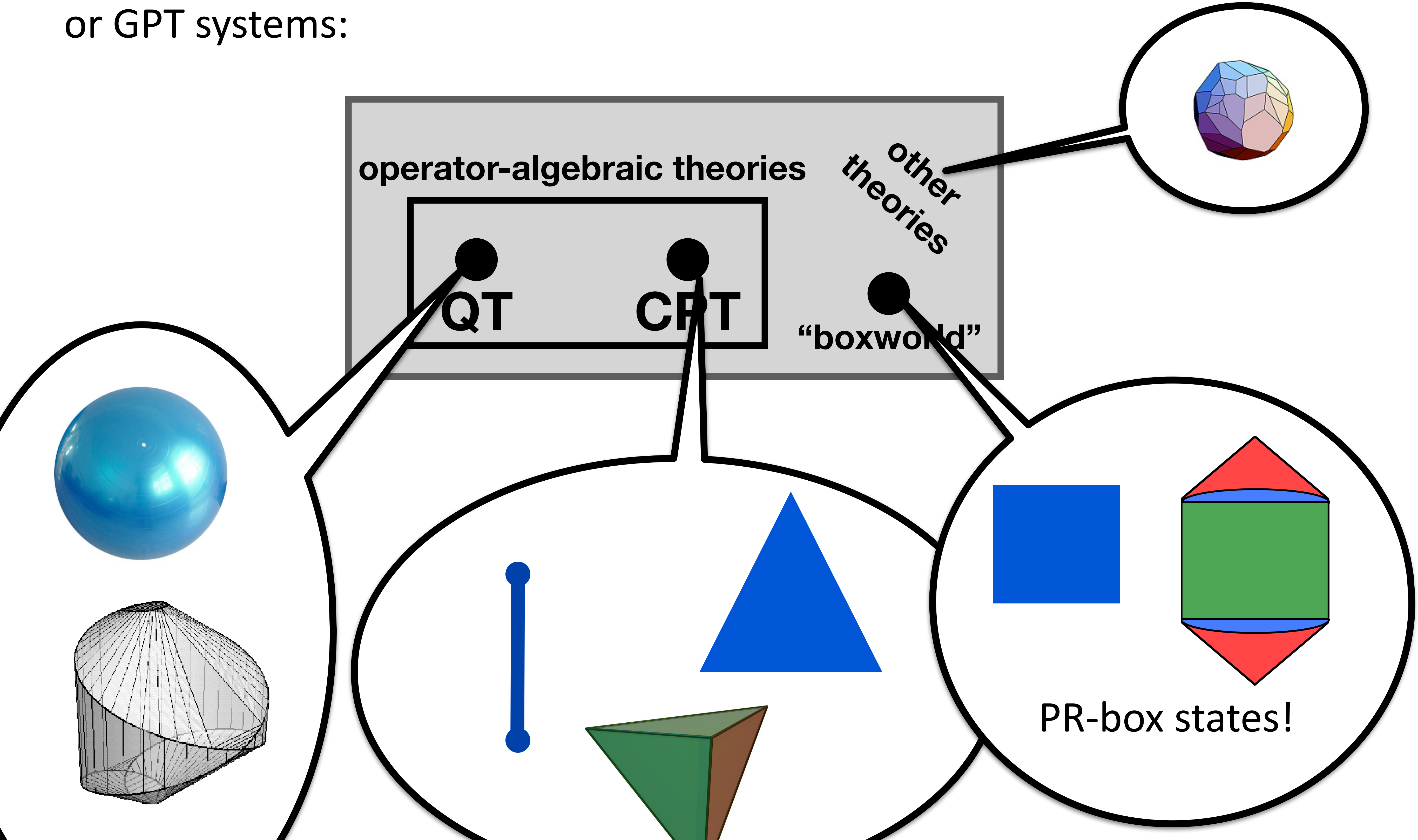

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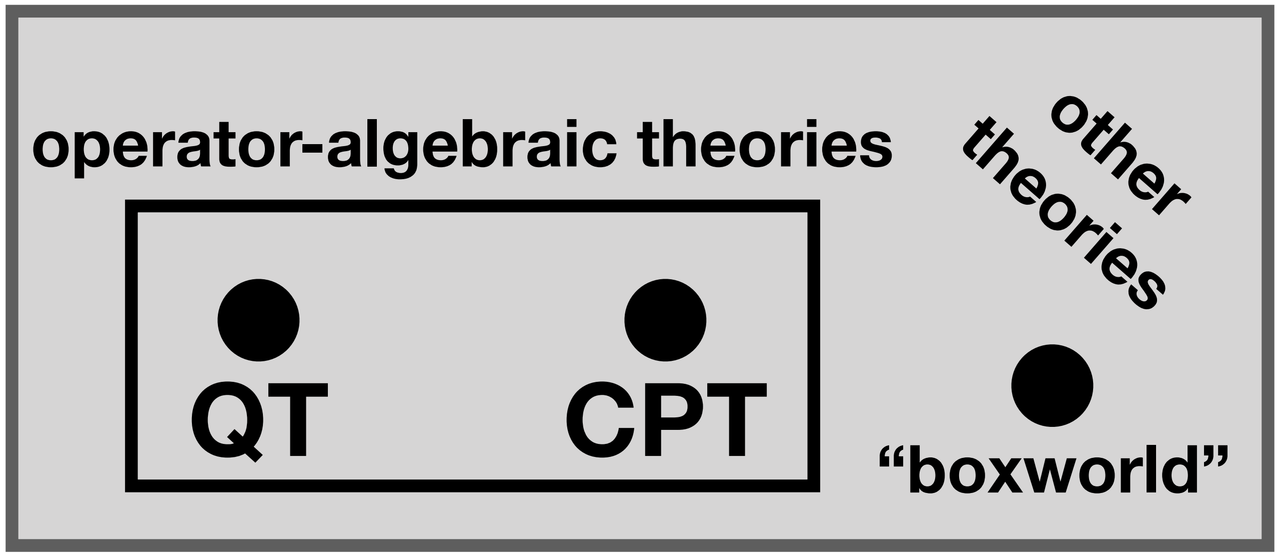

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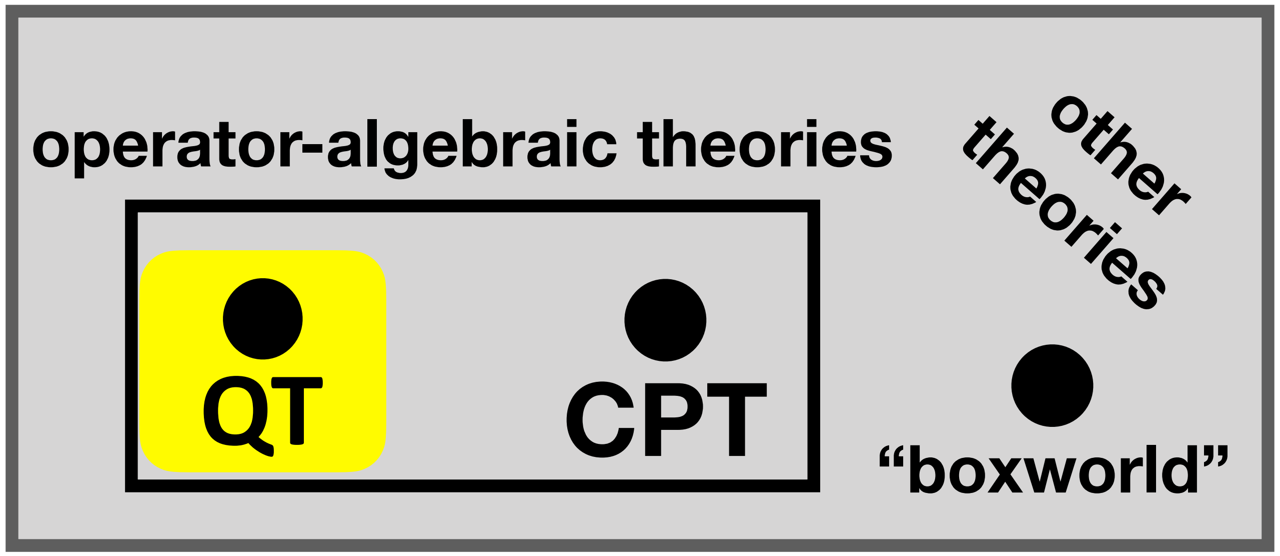

Goal: Find a small set of simple physical / information-theoretic **principles** that singles out QT uniquely.

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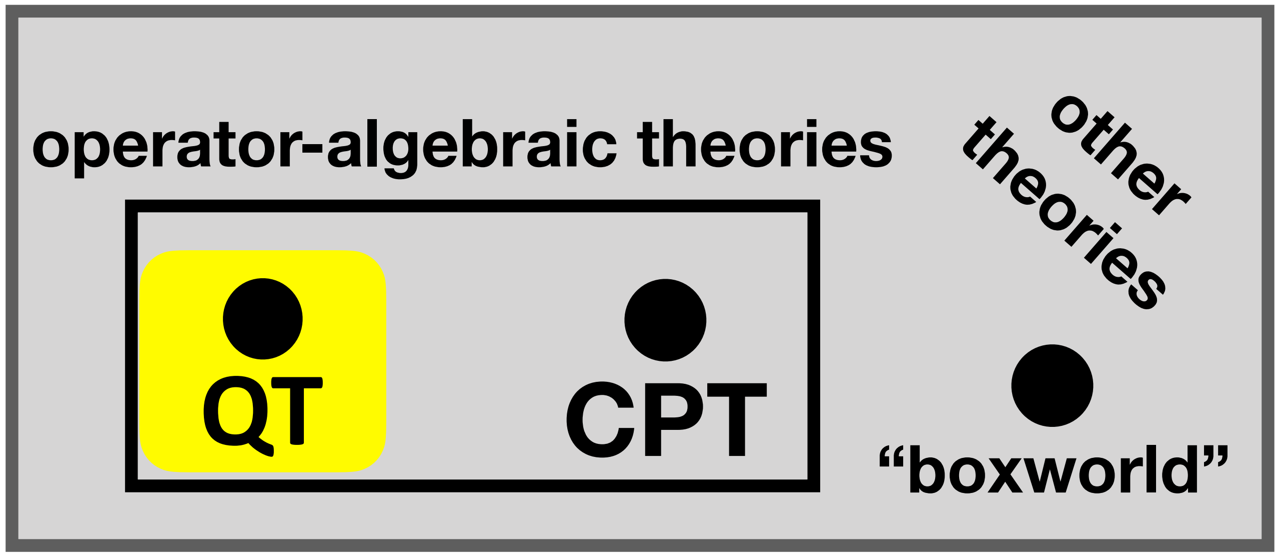

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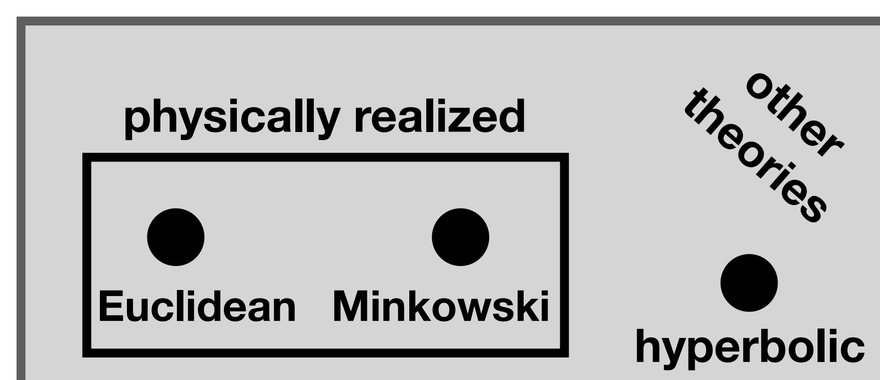
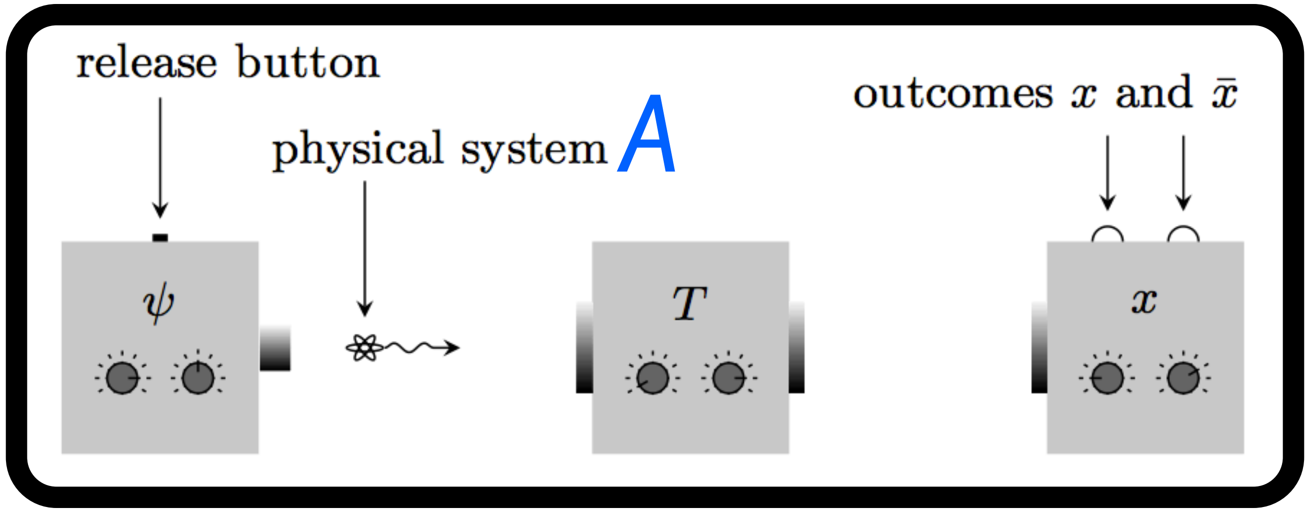
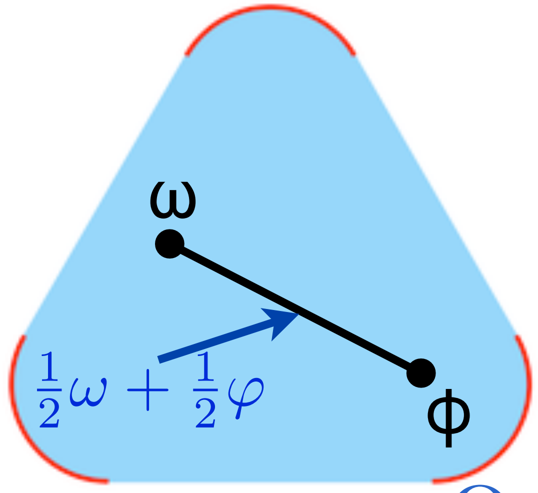
Role model: Einstein's **Relativity Principle** and **Light Postulate** determine Minkowski spacetime.



QT from three principles ("axioms")





Normalized state space Ω_A

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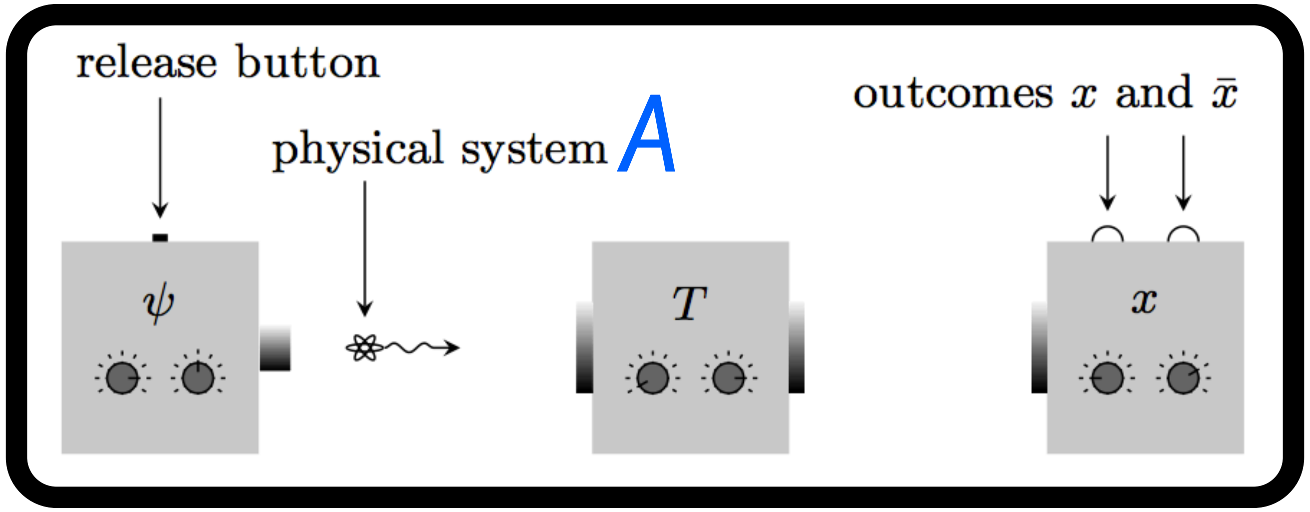
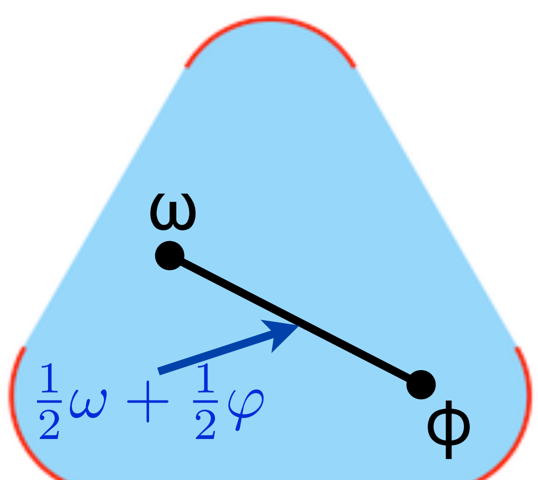


Transformations T map (unnormalized) states to states, and are linear.



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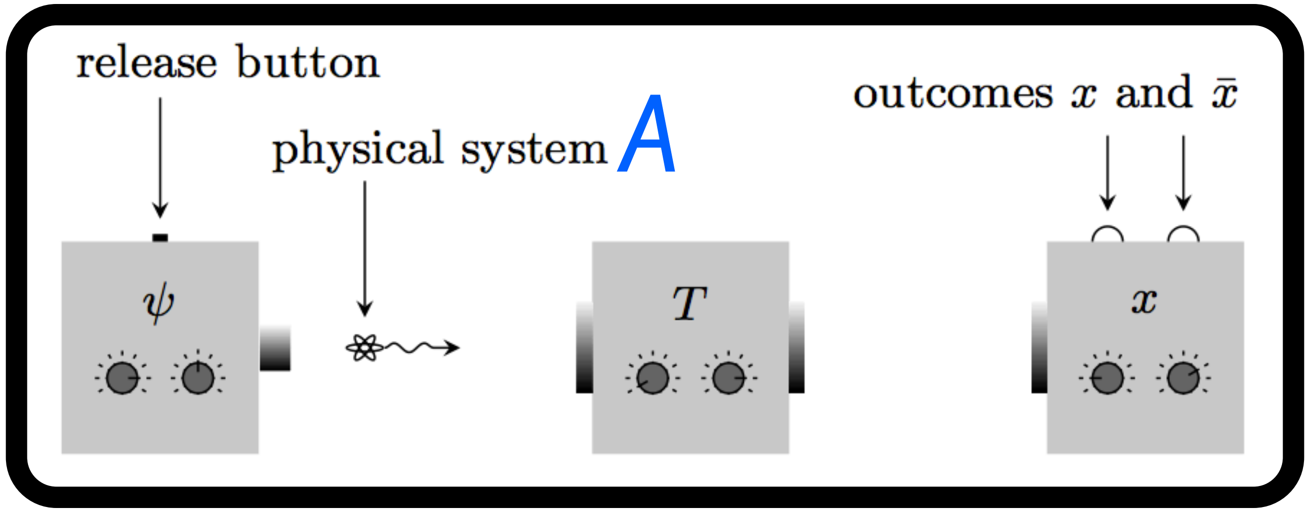

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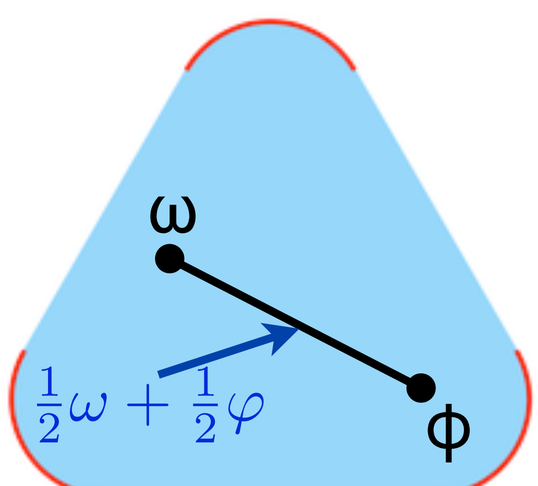
They are symmetries of state space: $T(\Omega_A) = \Omega_A$



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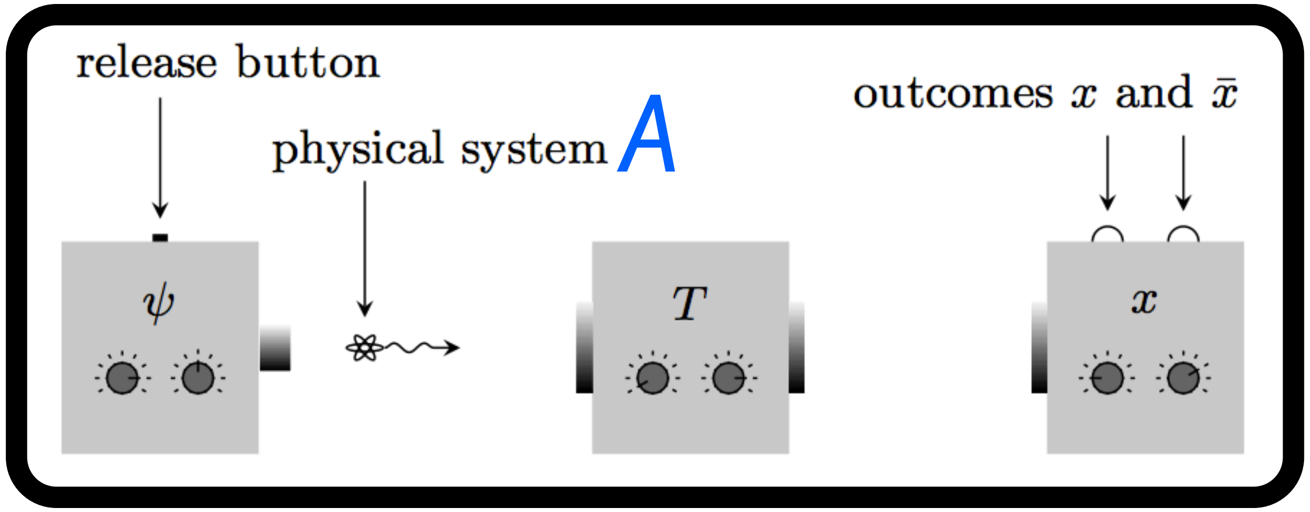

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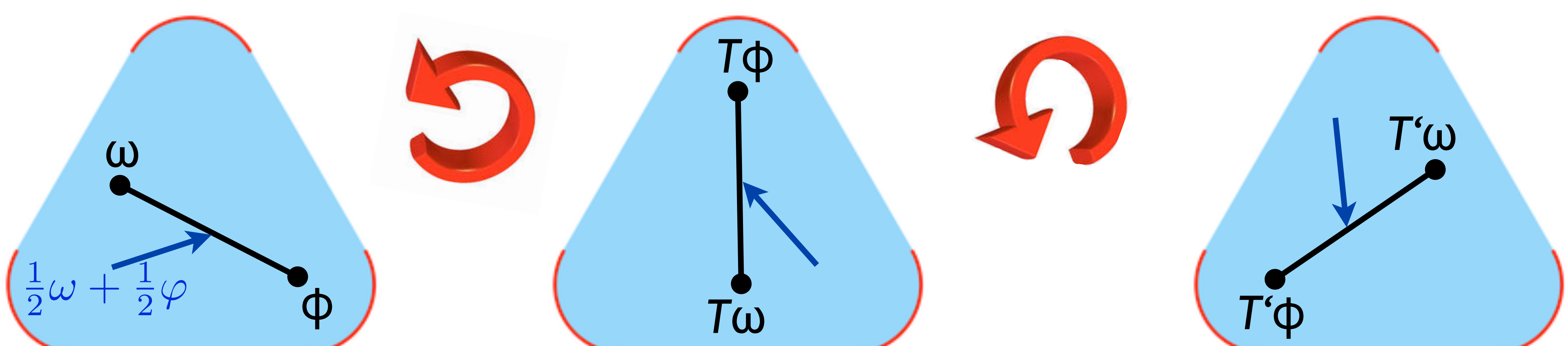

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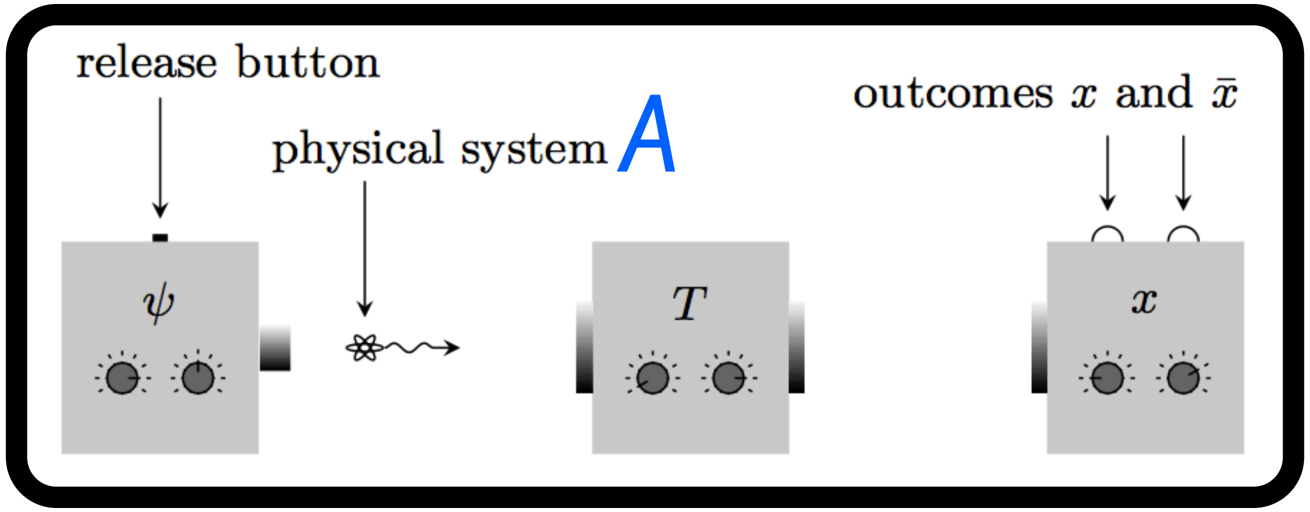

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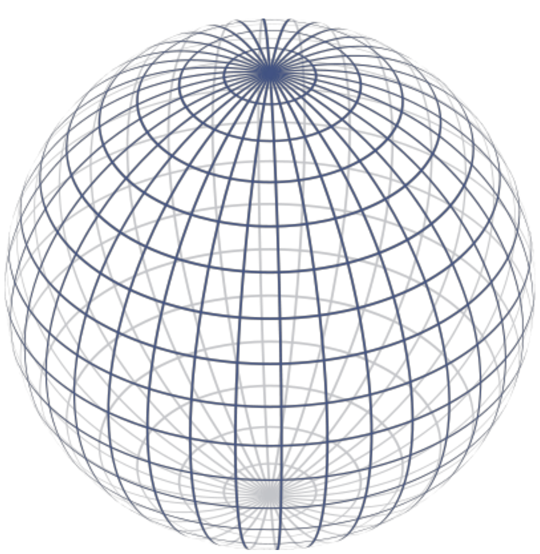

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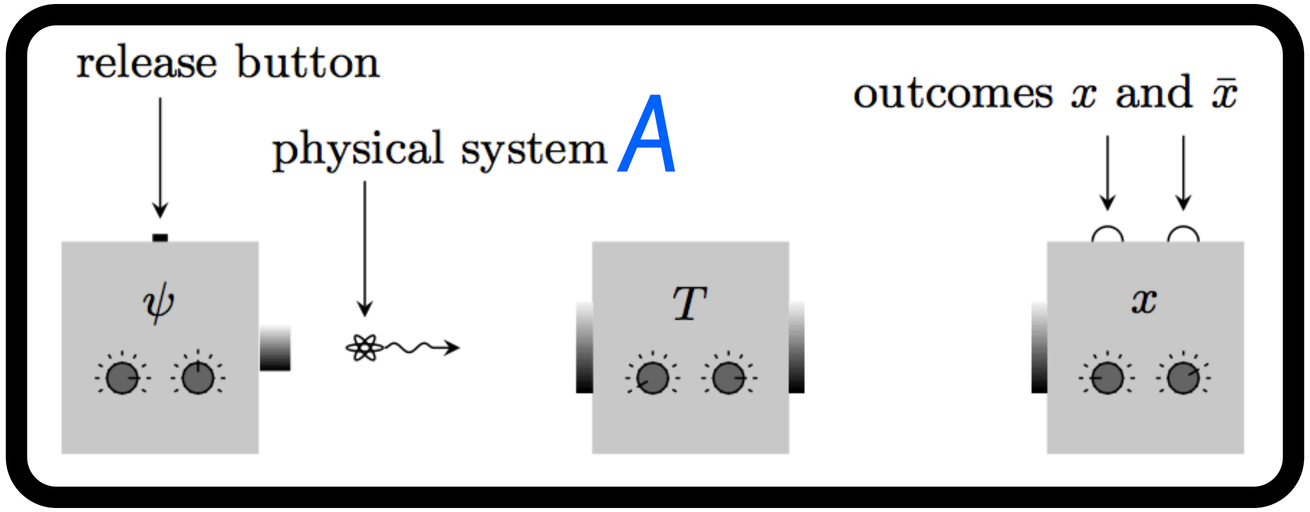

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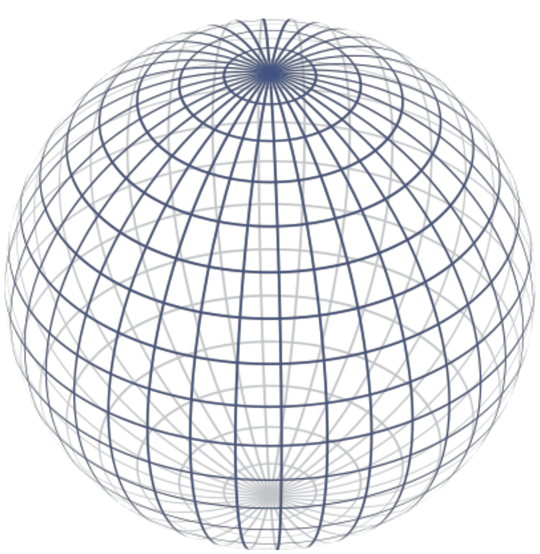

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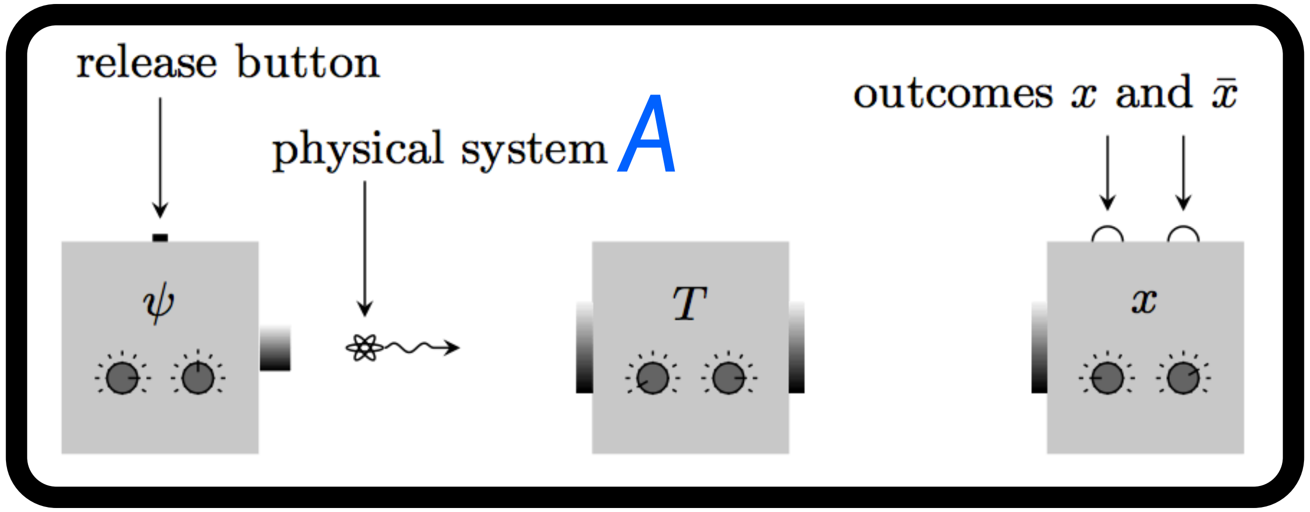
state space

reversible transformations

QT from three principles ("axioms")



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If ϕ and ω are pure, then there is a reversible T with $T\phi=\omega$.

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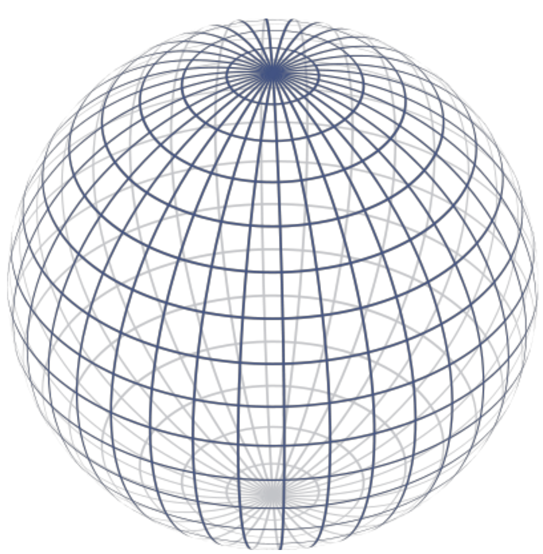

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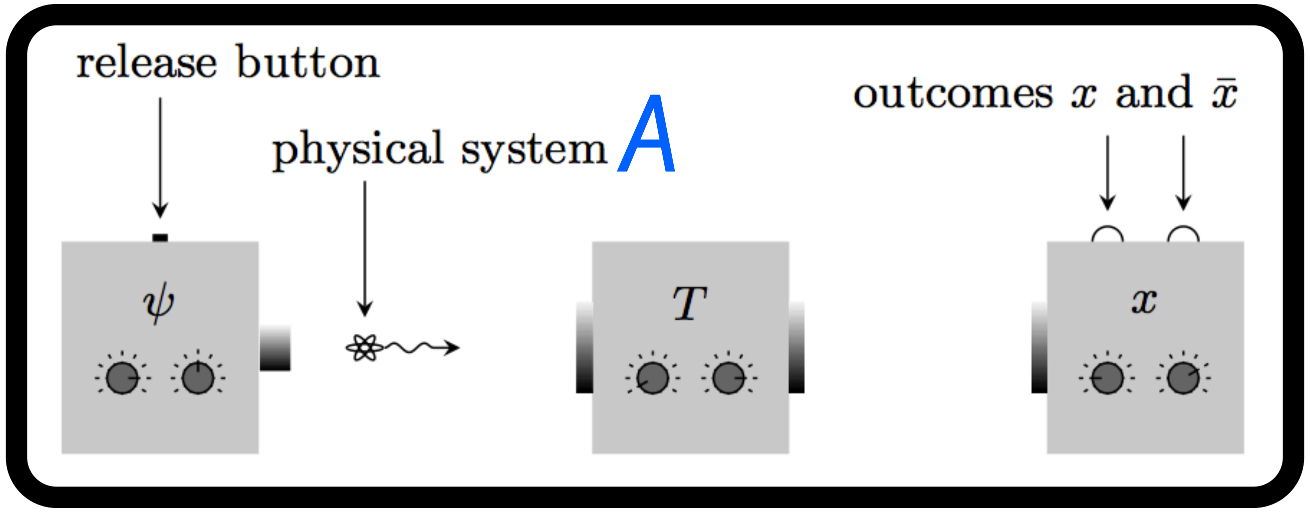

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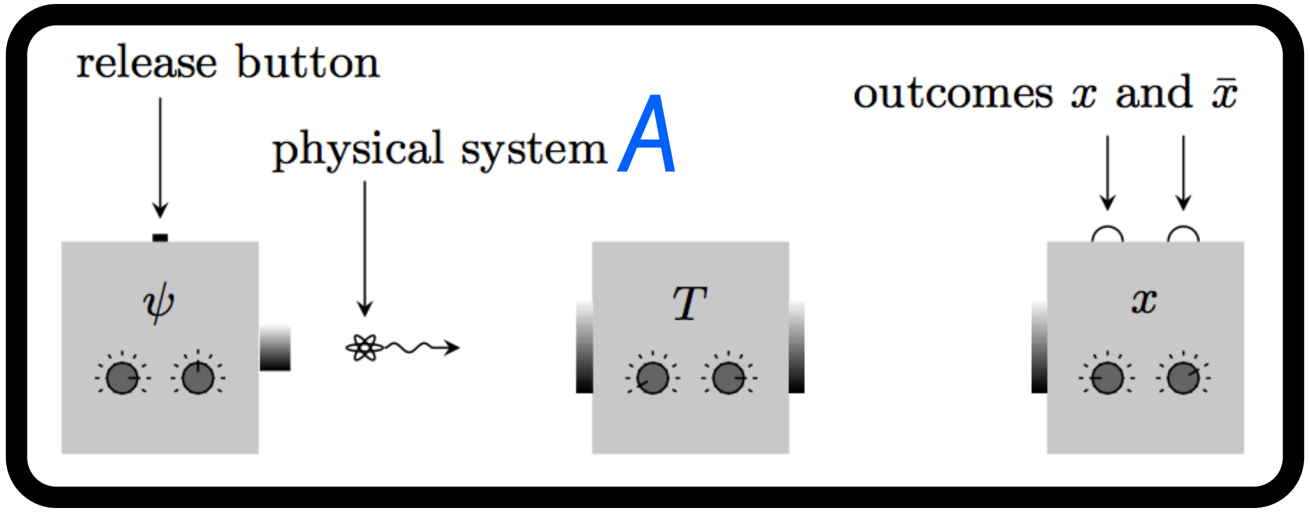

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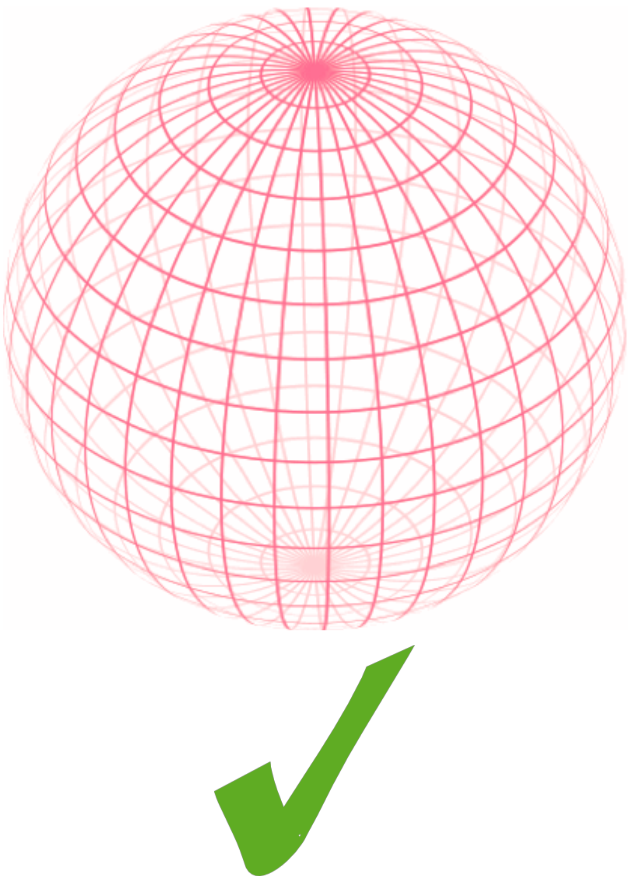

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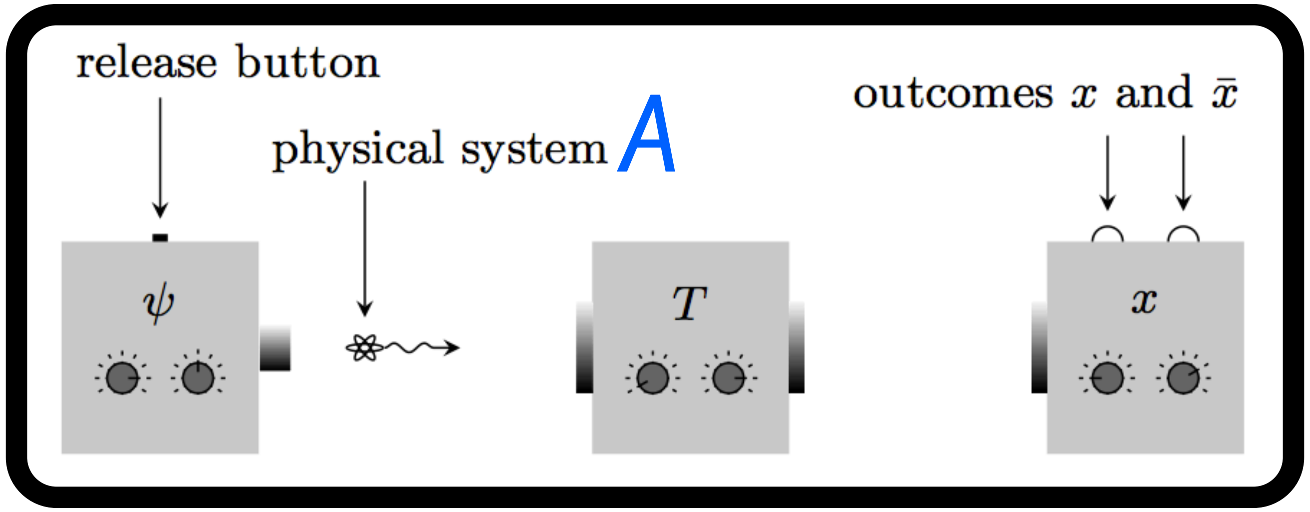

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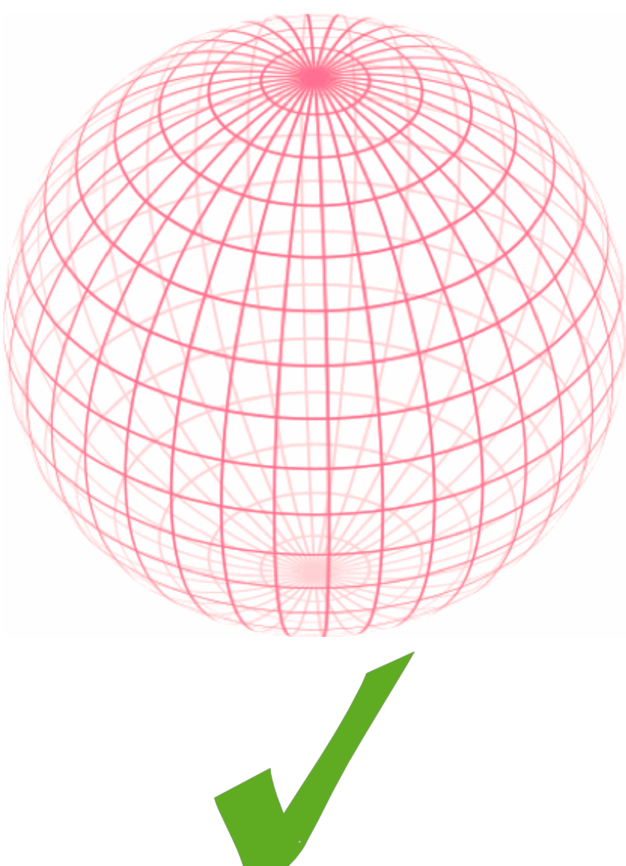

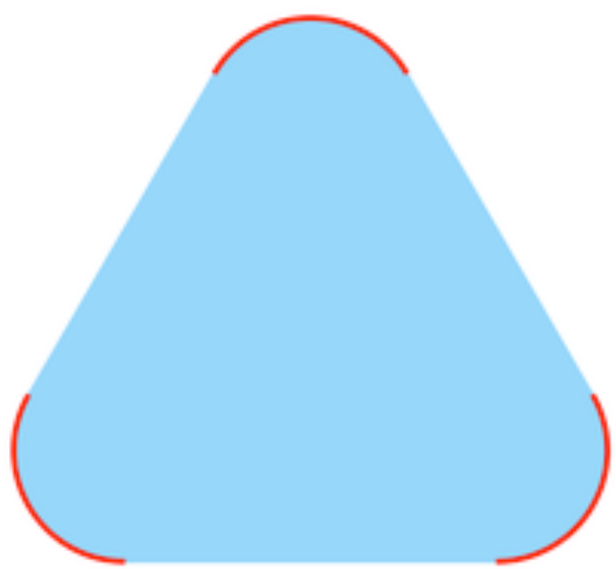

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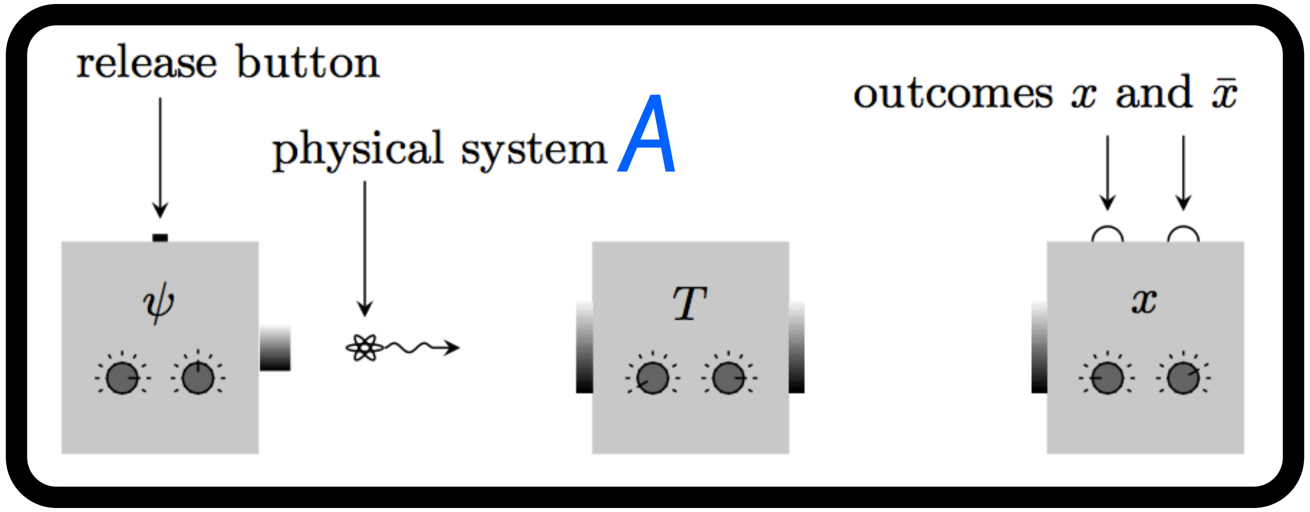
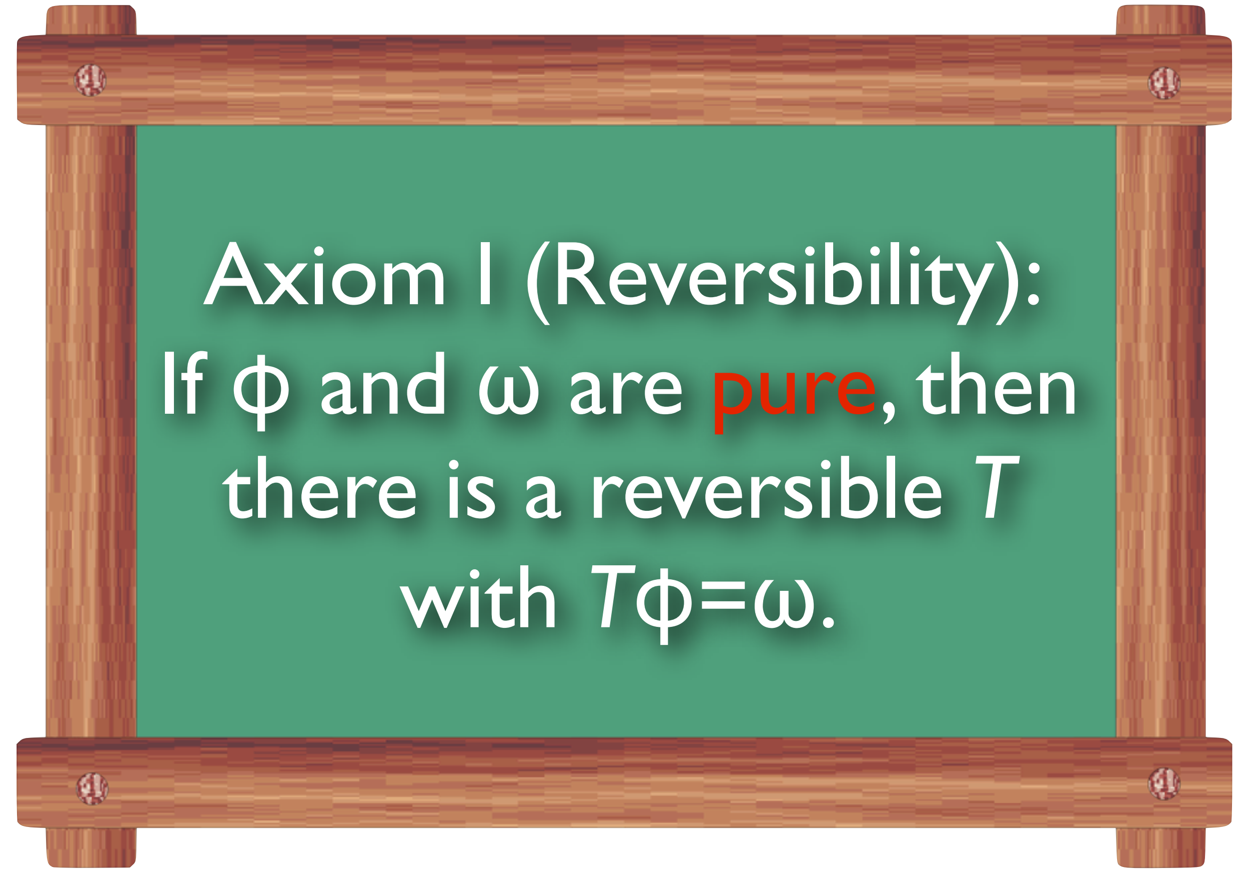
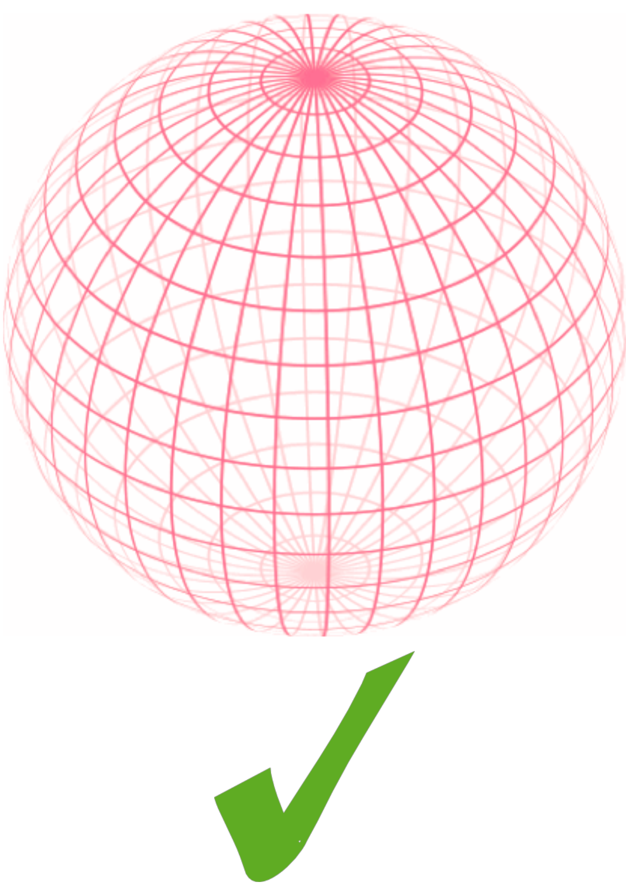
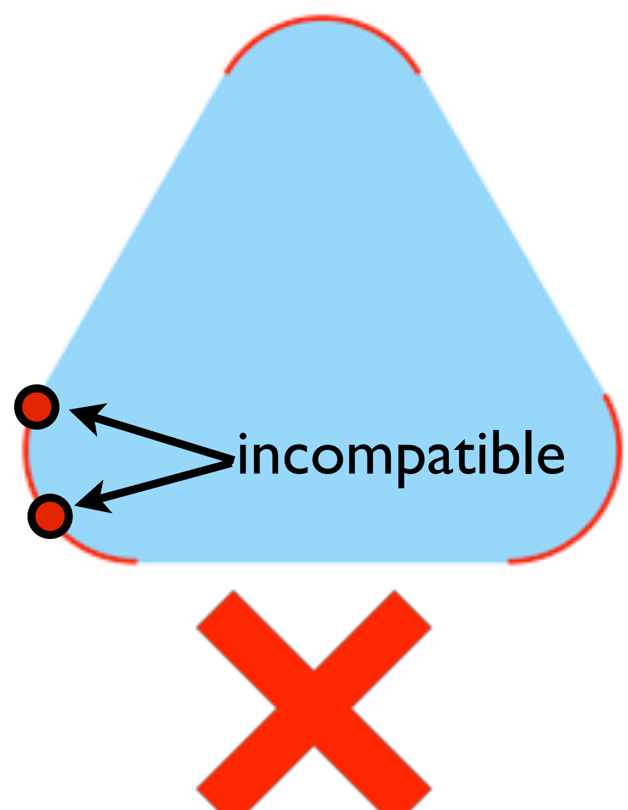




Enforces some symmetry in state space:





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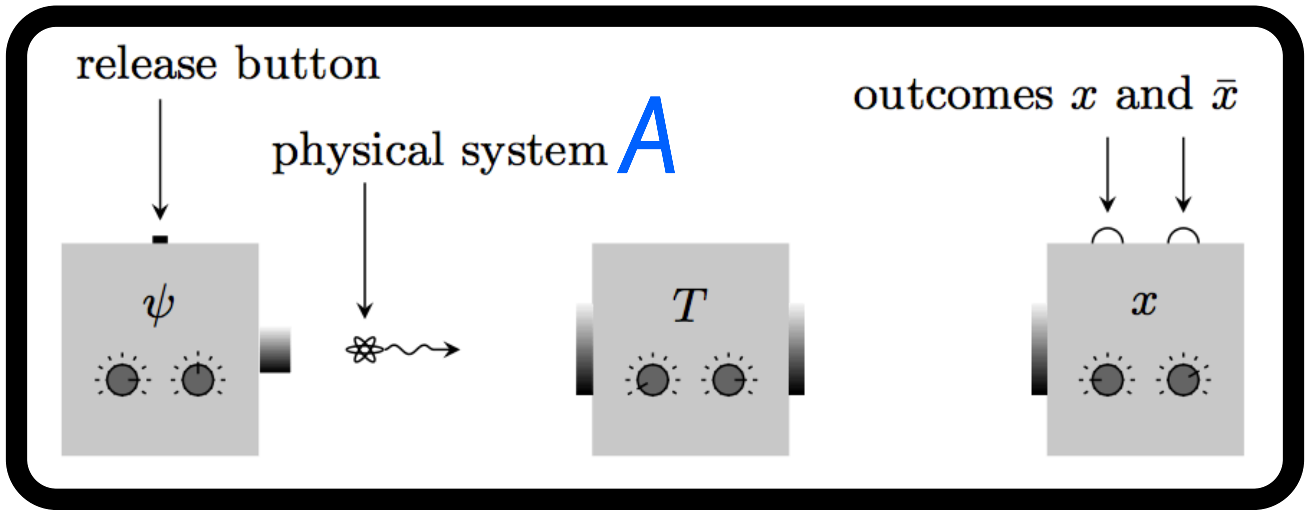

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If ϕ and ω are pure, then there is a reversible T with $T\phi=\omega$.

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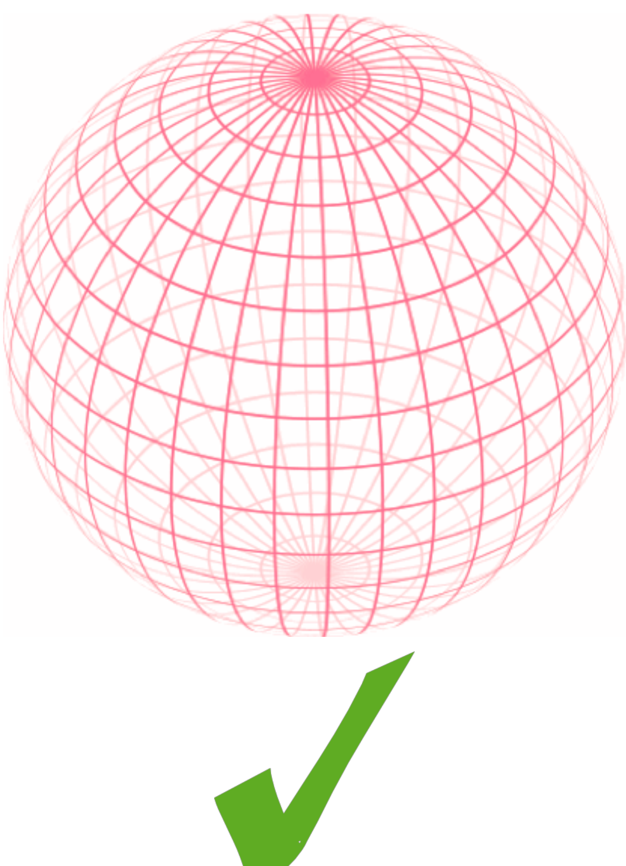

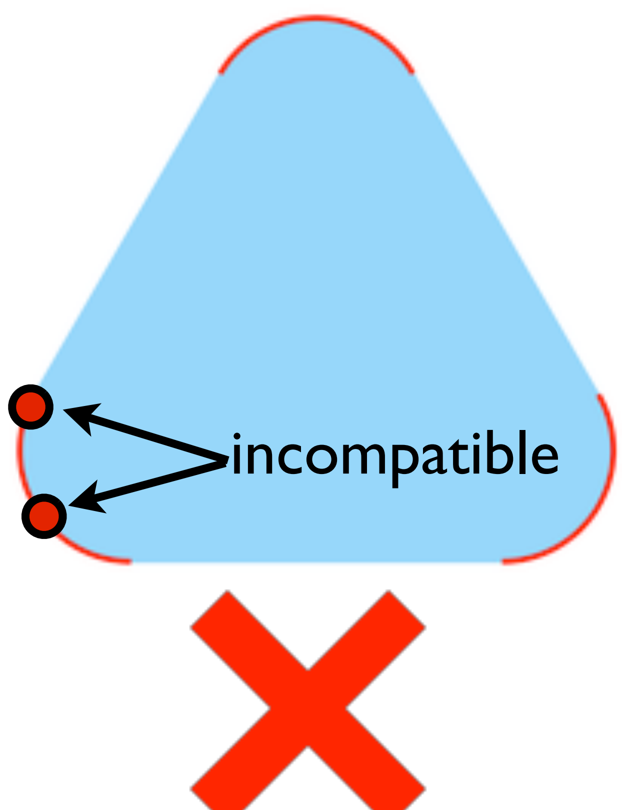

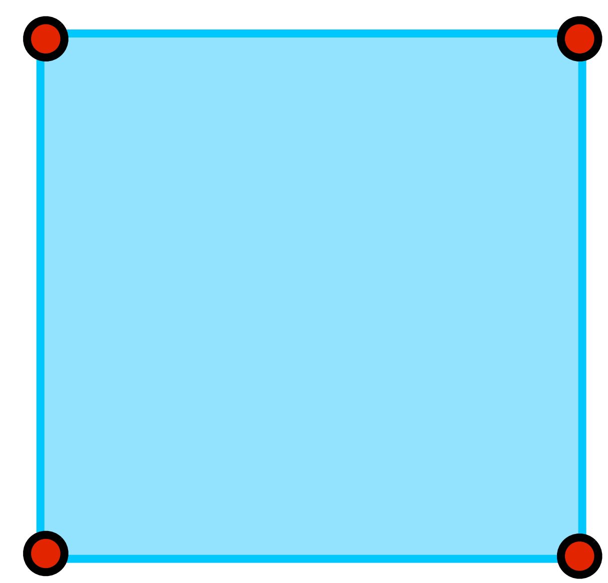

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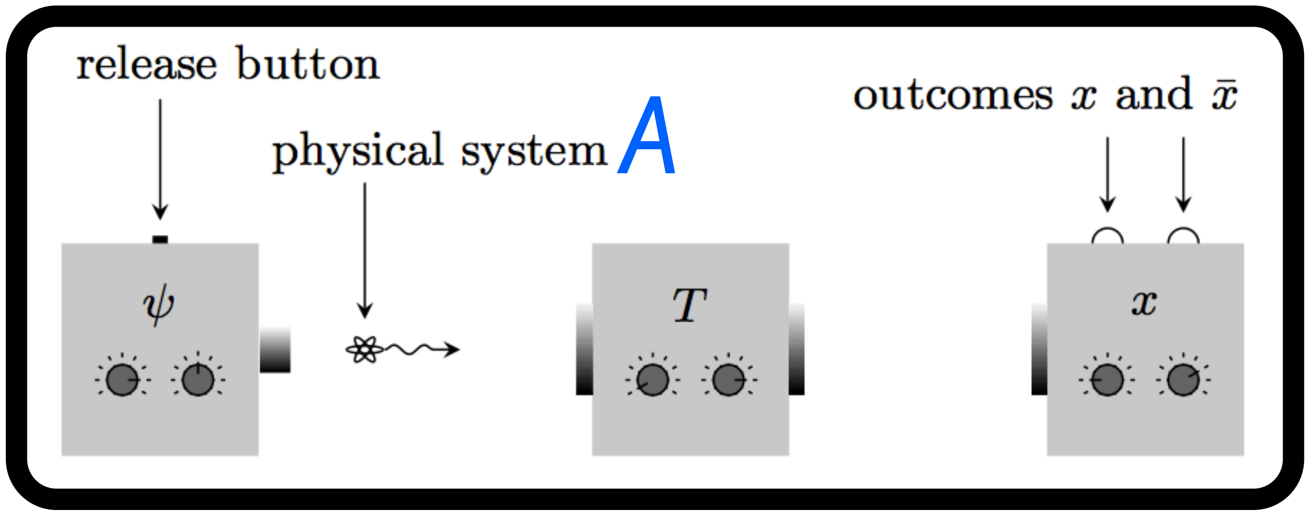

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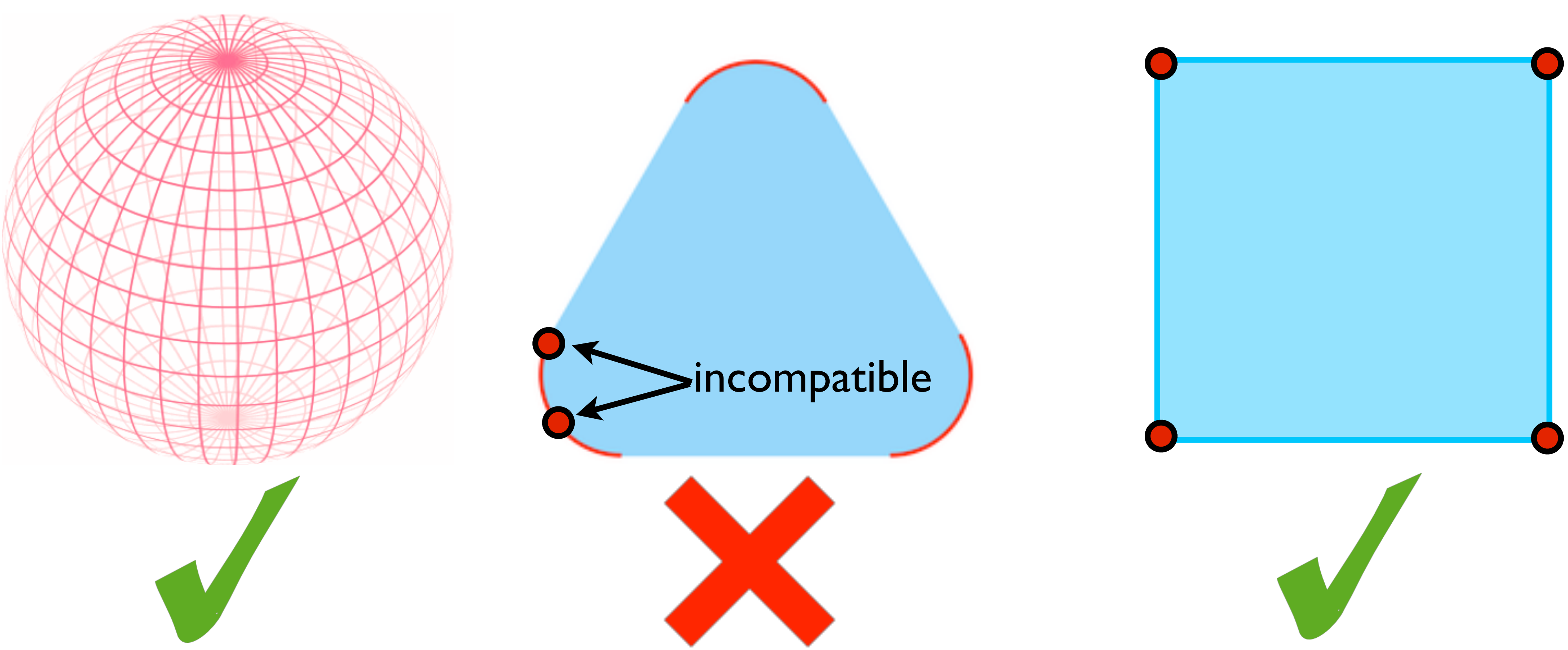

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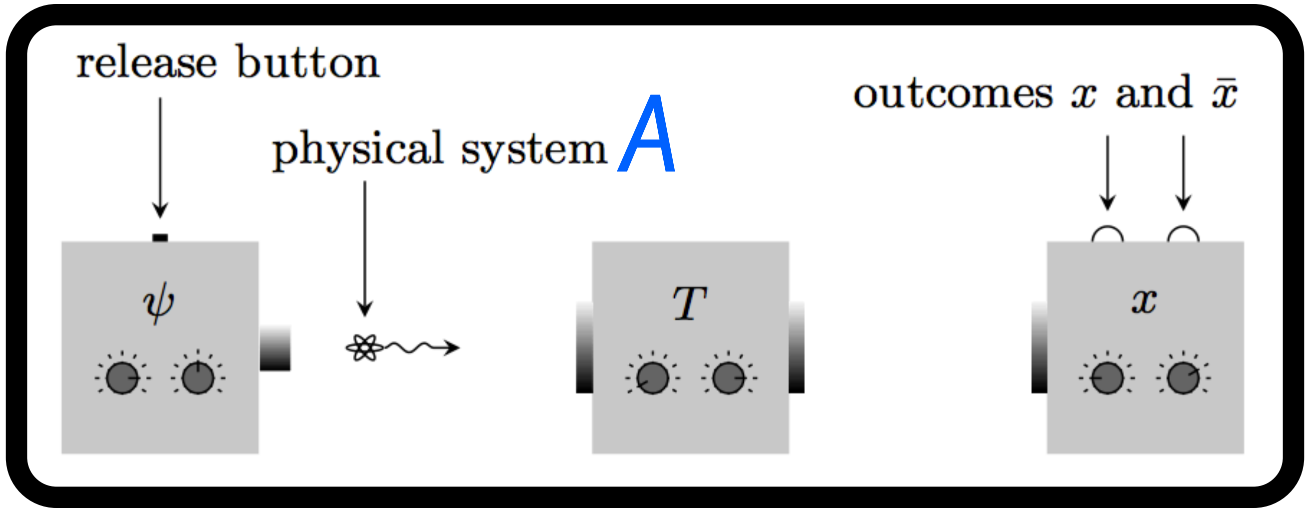

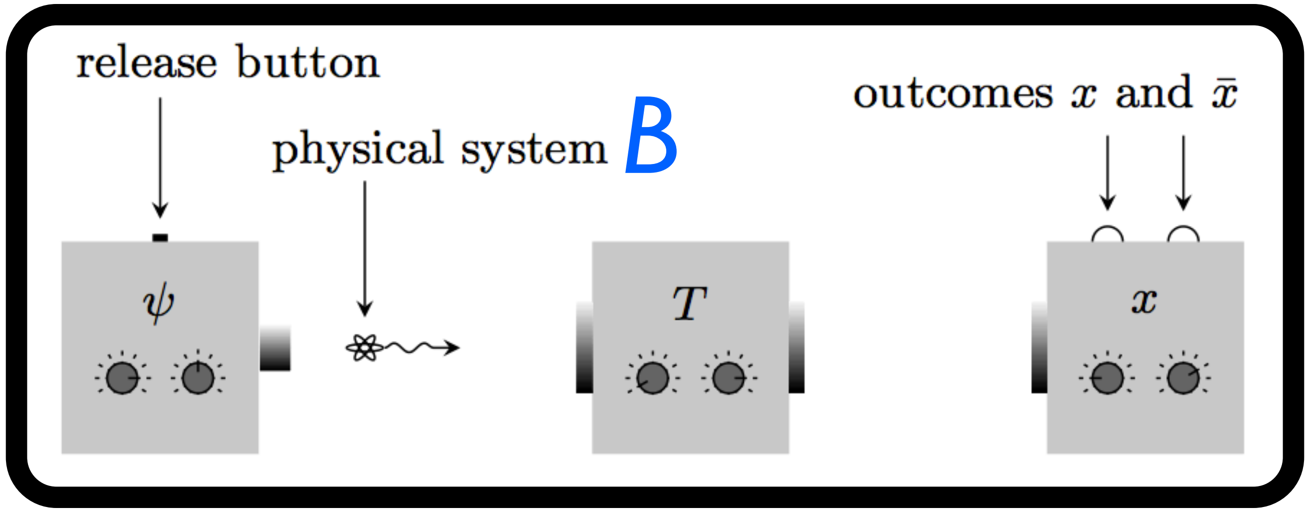

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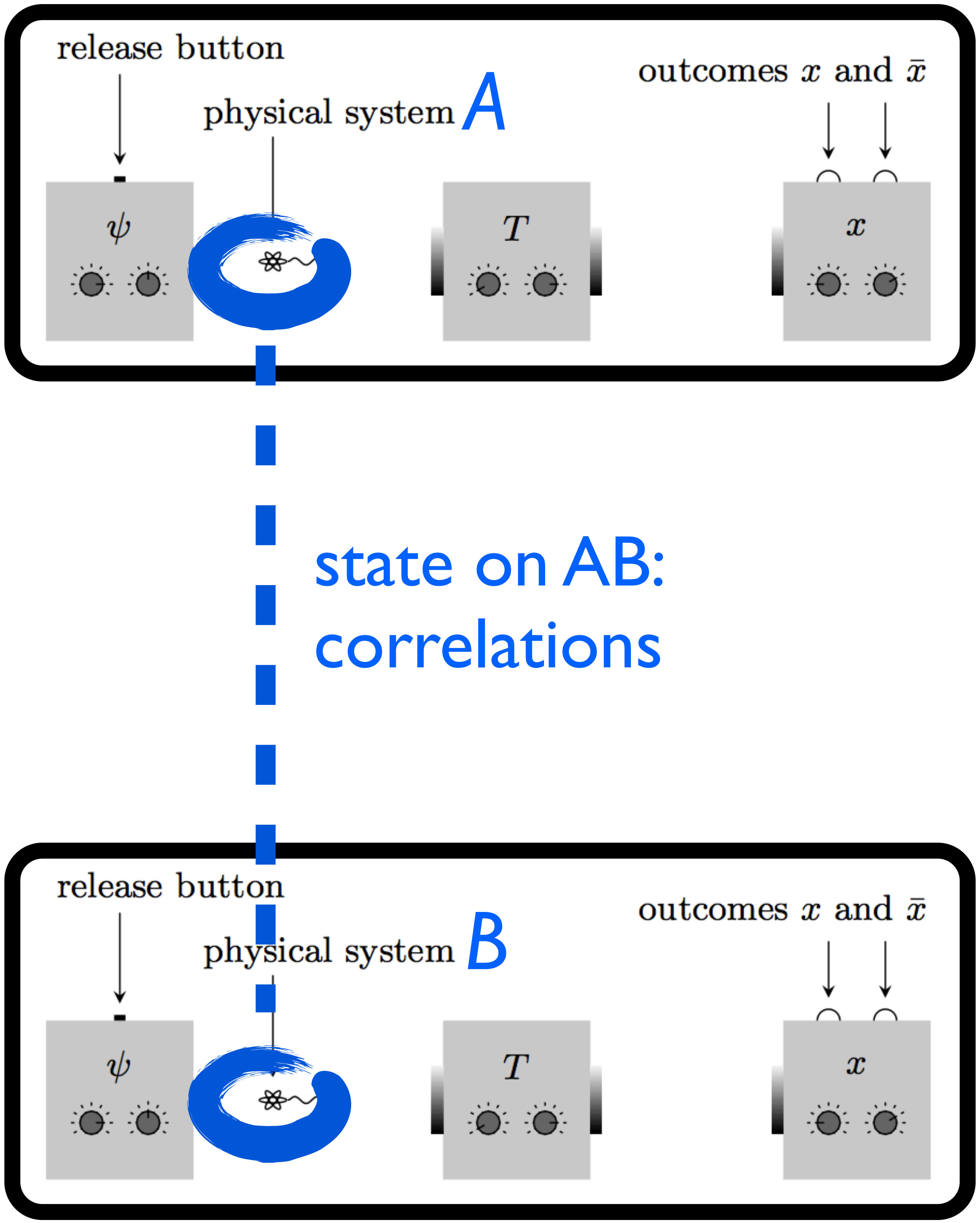

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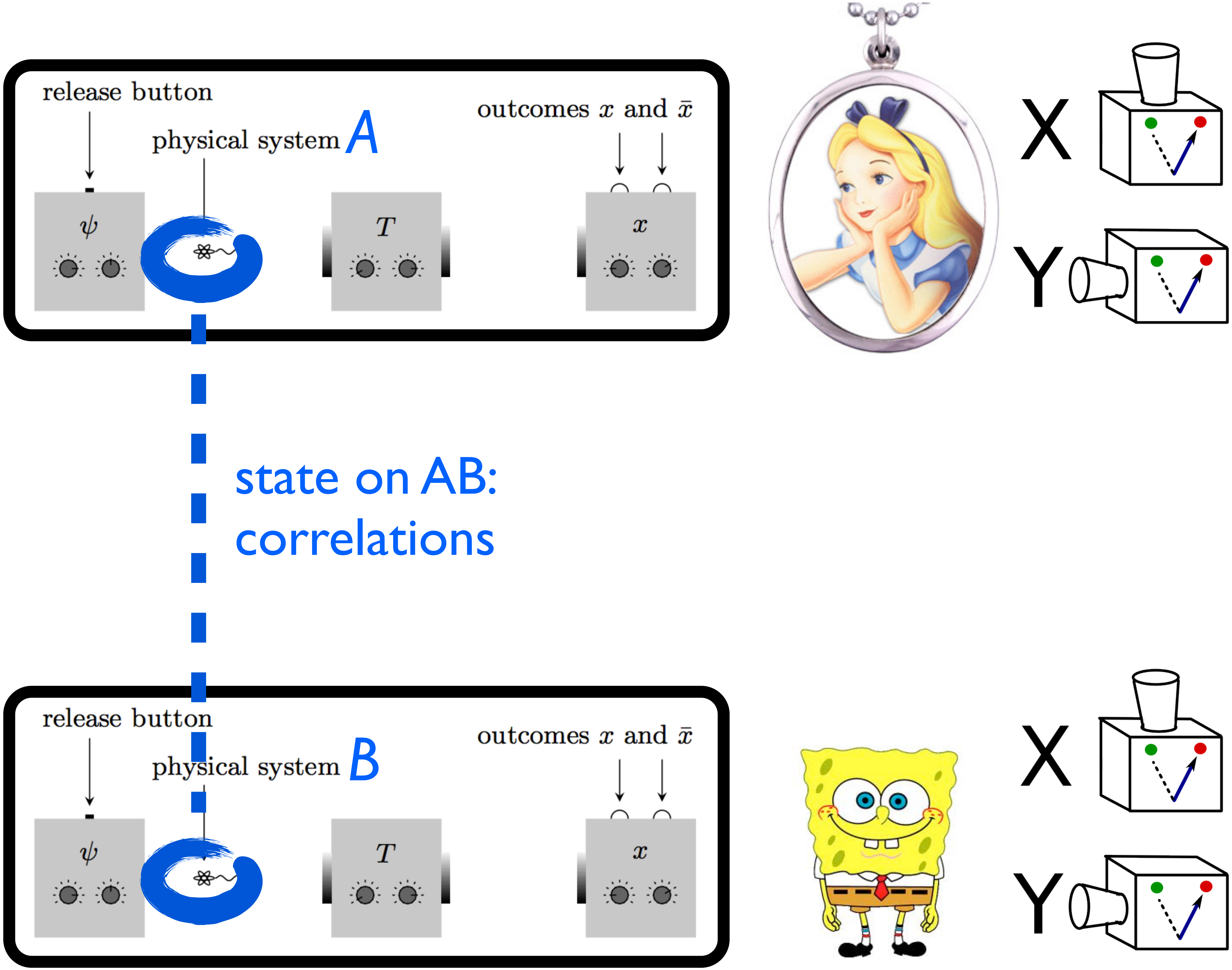

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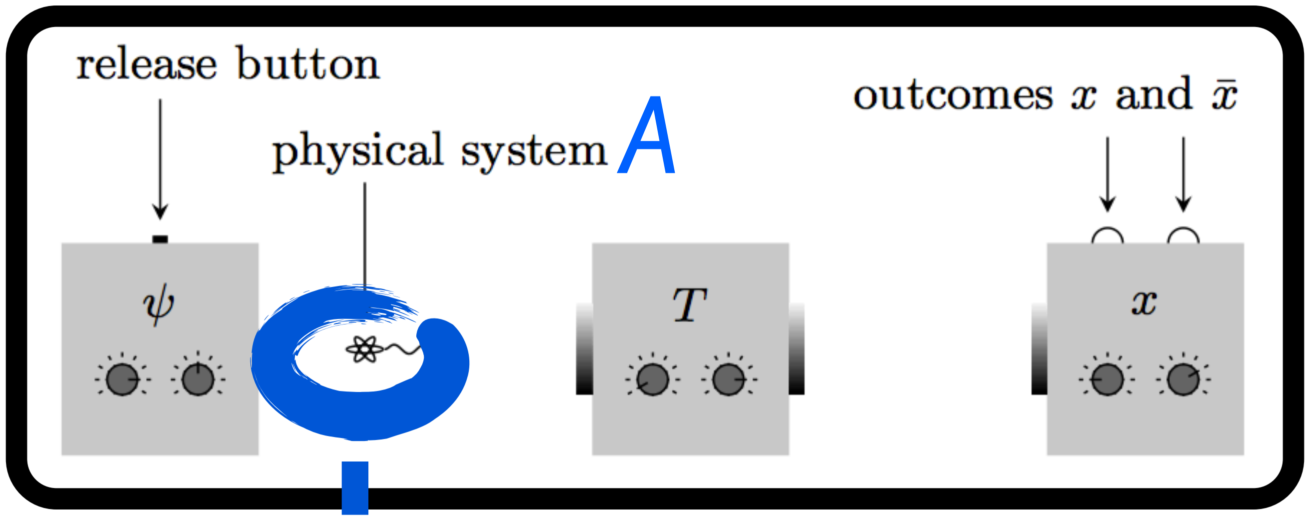
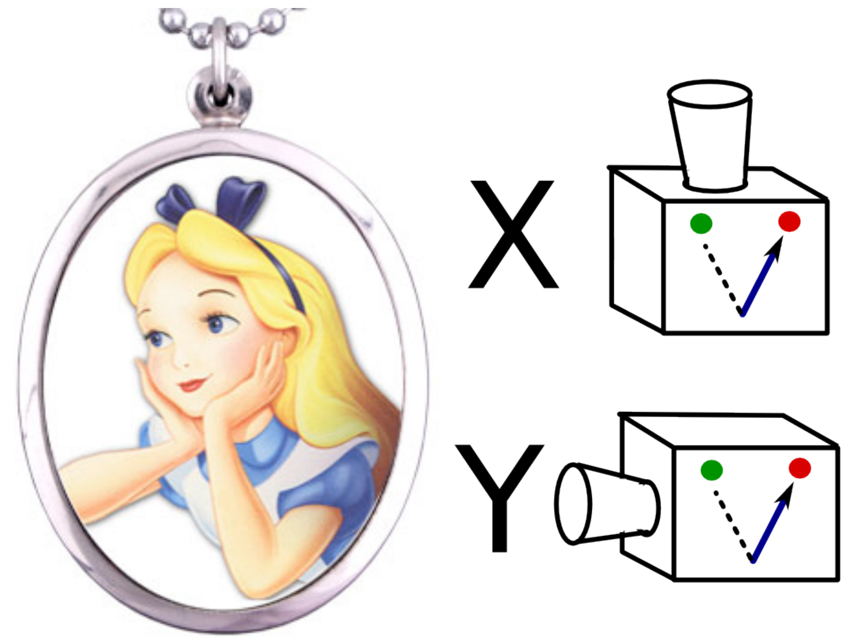
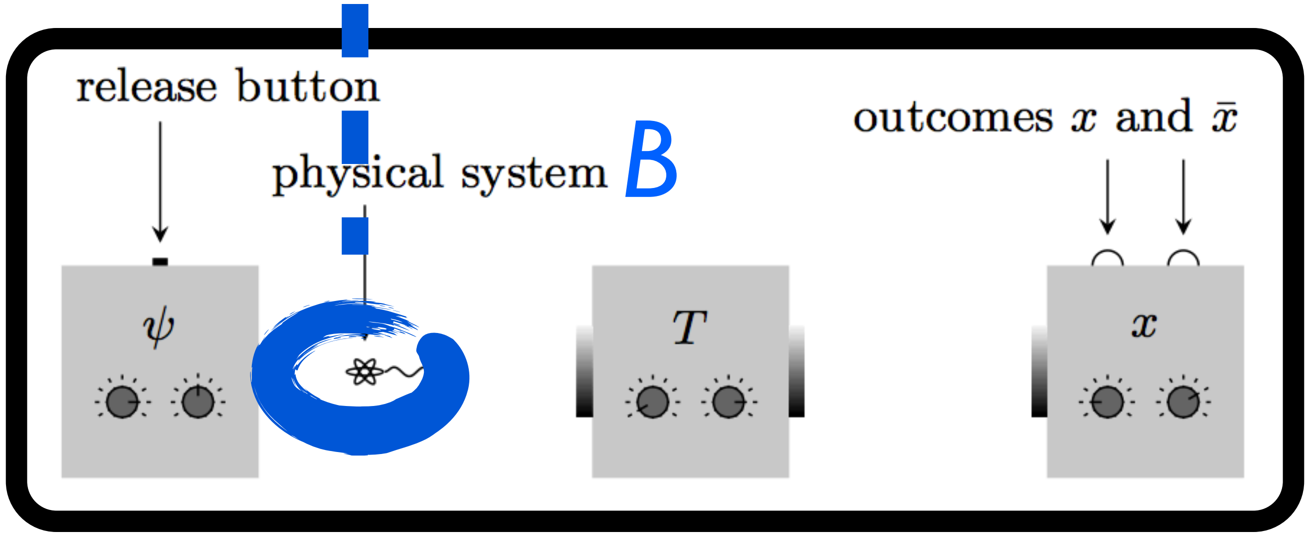
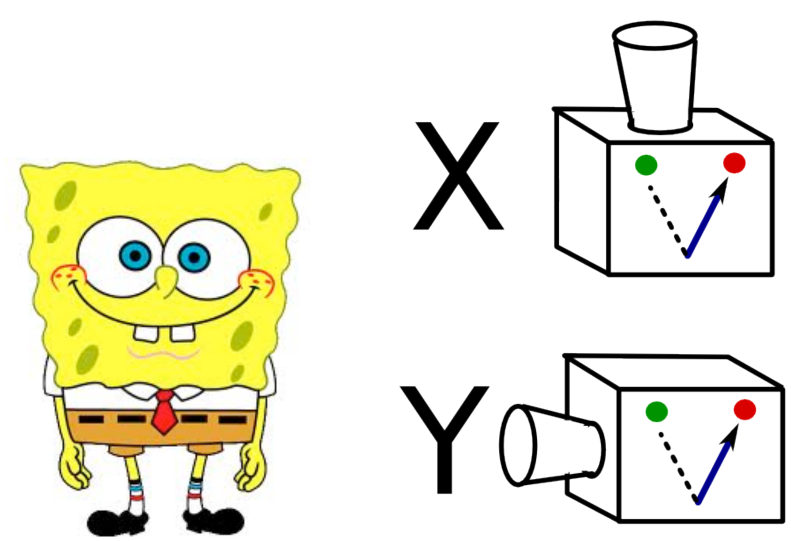




state on AB:
correlations

No-signalling condition:
Alice's probabilities do not depend on Bob's choice of measurement.





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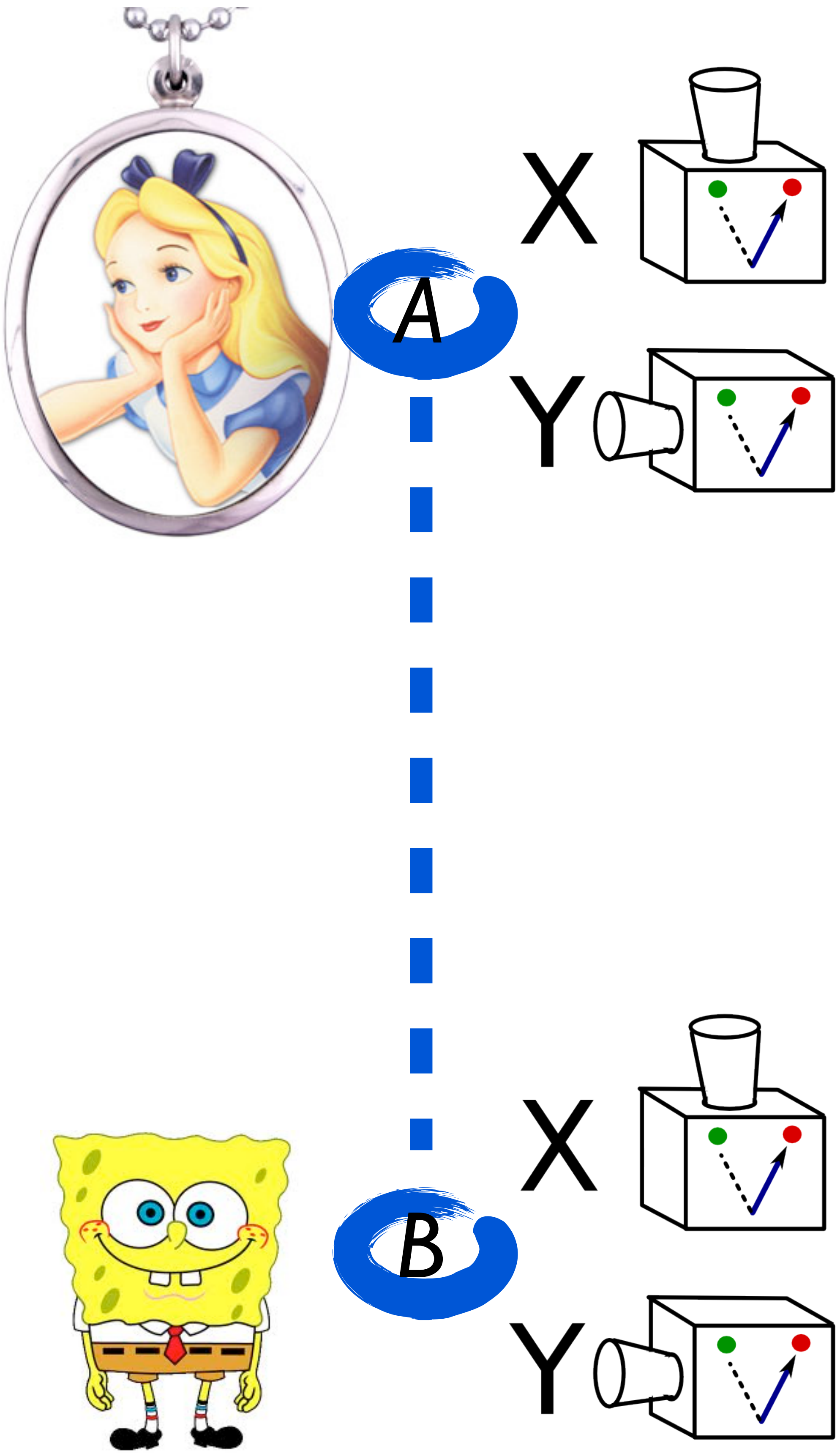

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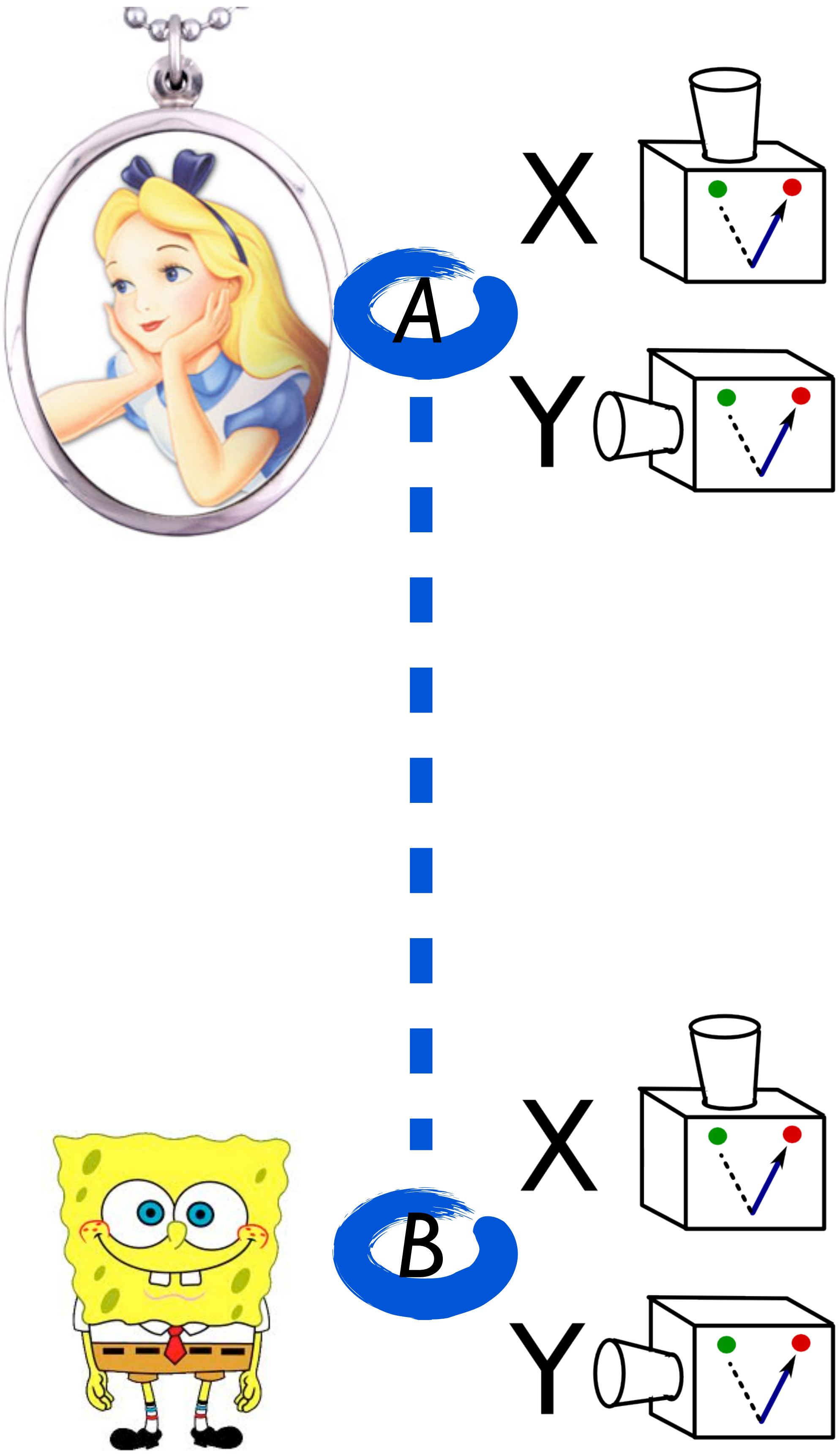

Axiom II: States on AB are uniquely determined by <u>correlations of local measurements</u> on A,B.

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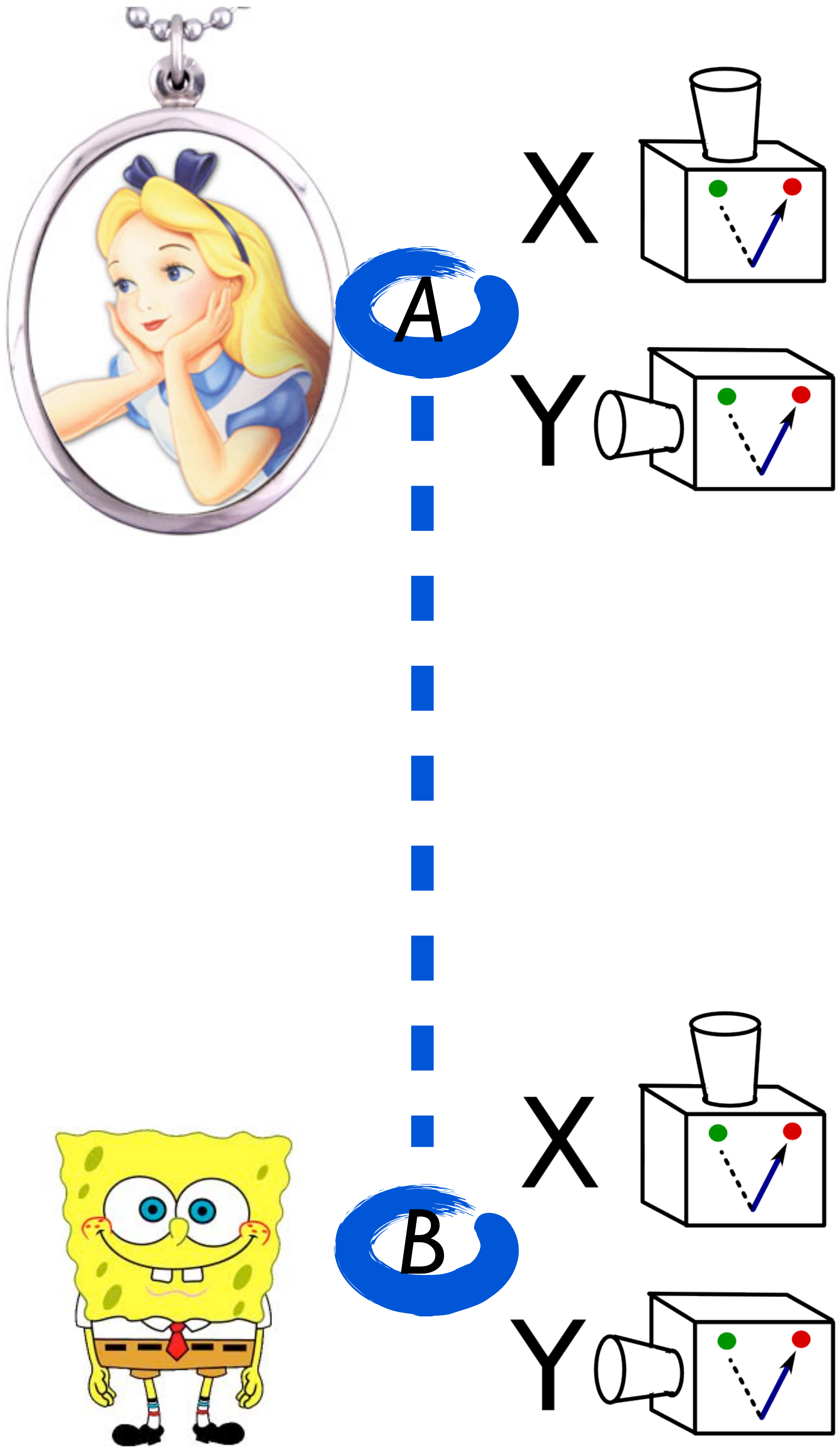

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= „Local tomography": No non-local measurements necessary.

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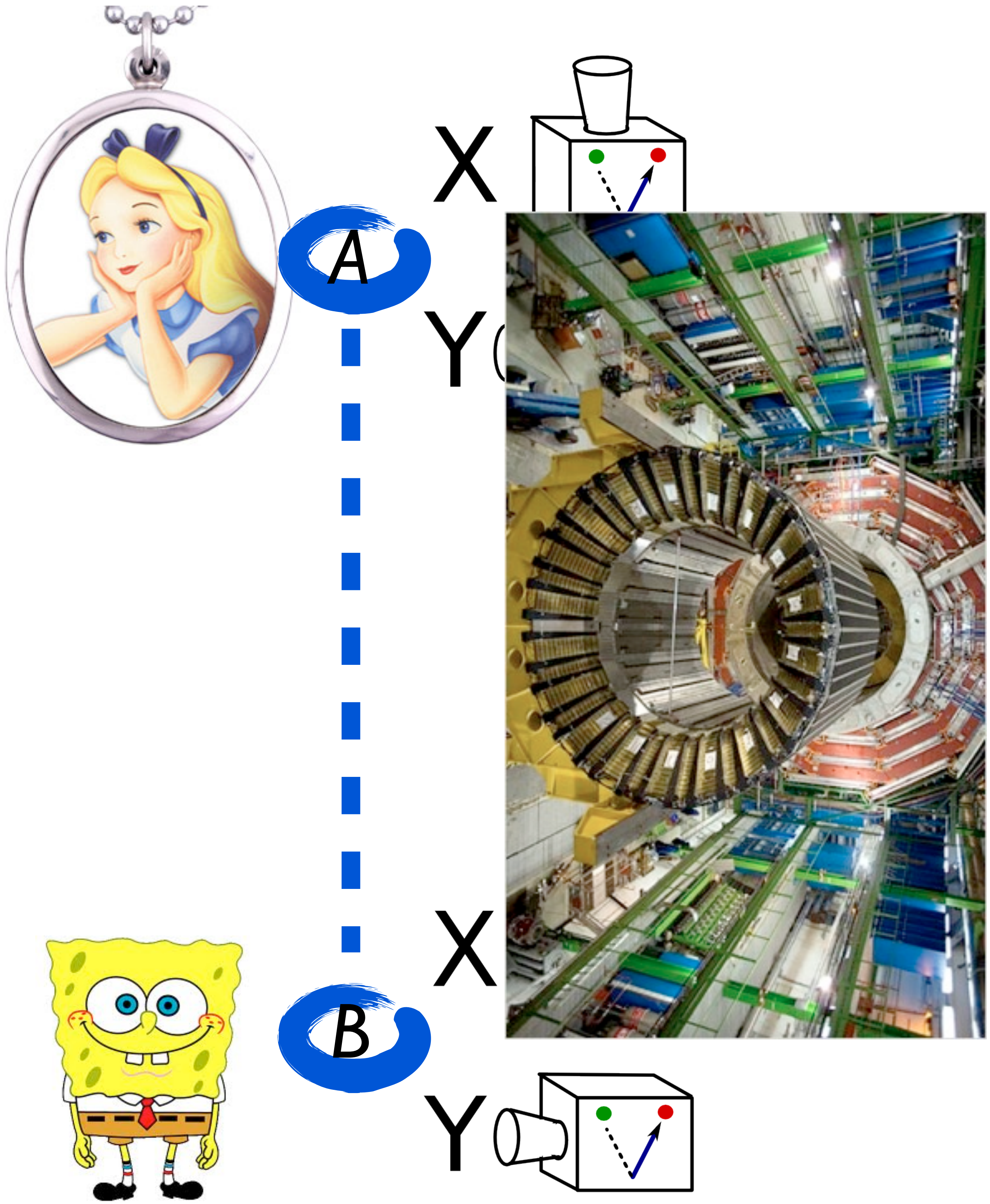

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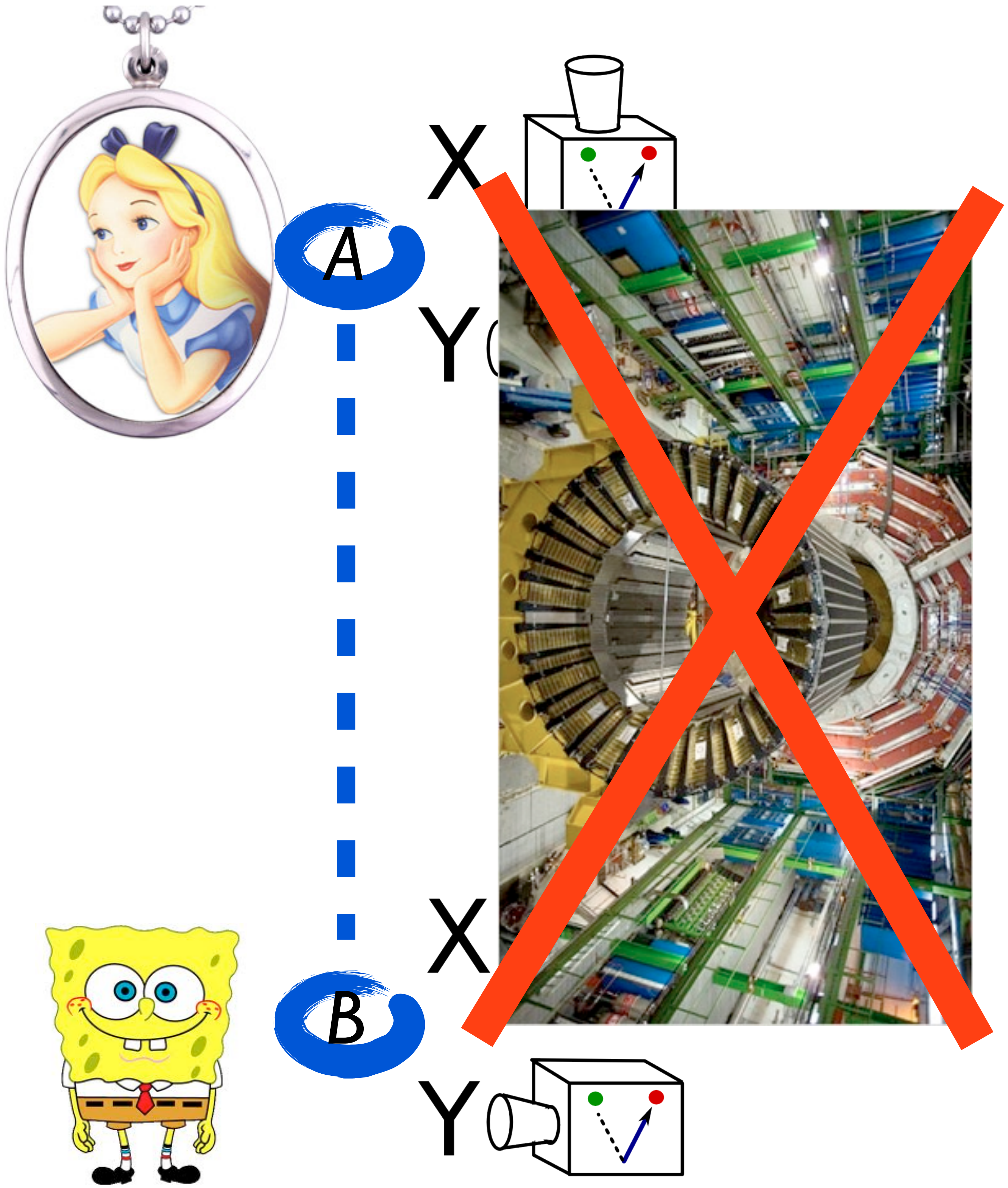

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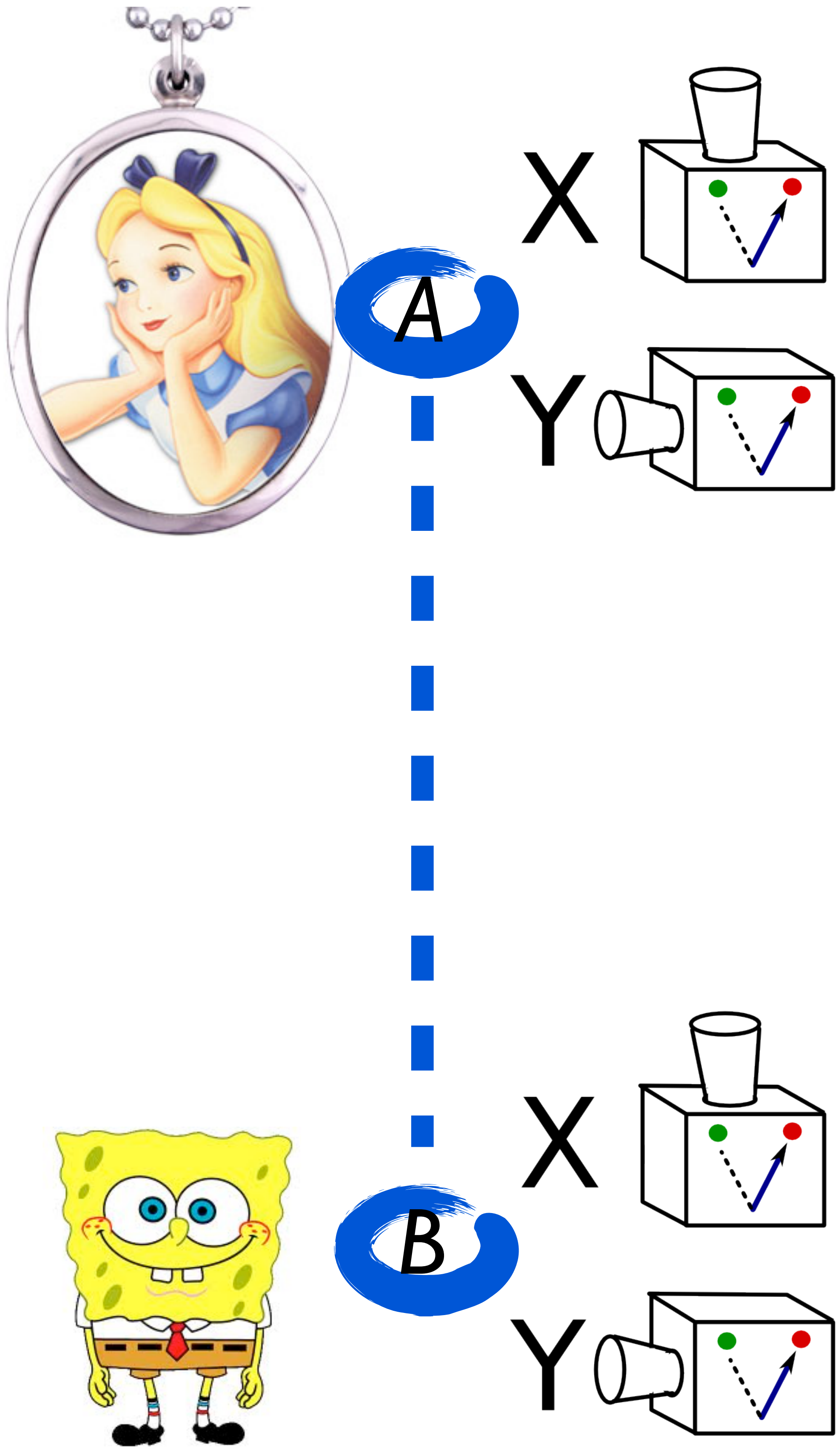

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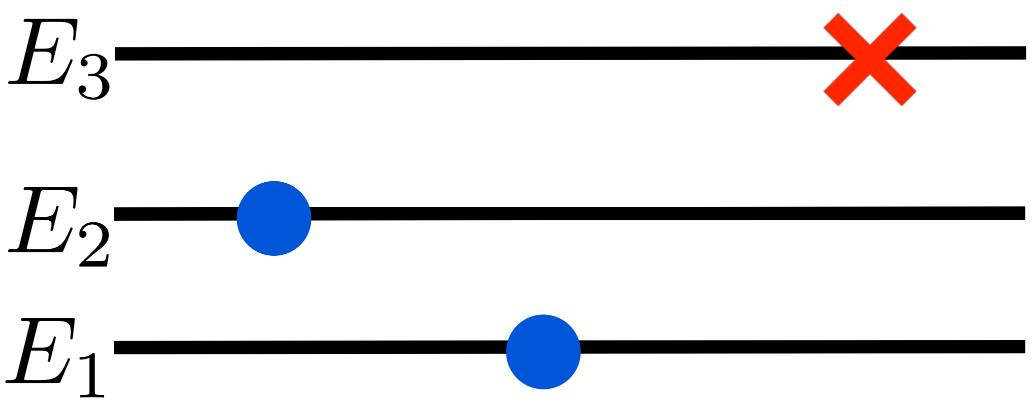
= „Local tomography“: No non-local measurements necessary.

Global state space $\Omega_{AB} \subset A \otimes B$ but not uniquely fixed!

Axiom III: The Subspace Axiom

Some 3-level system:



Impossible to have system in 3rd level
⇒ find particle there with probab. 0

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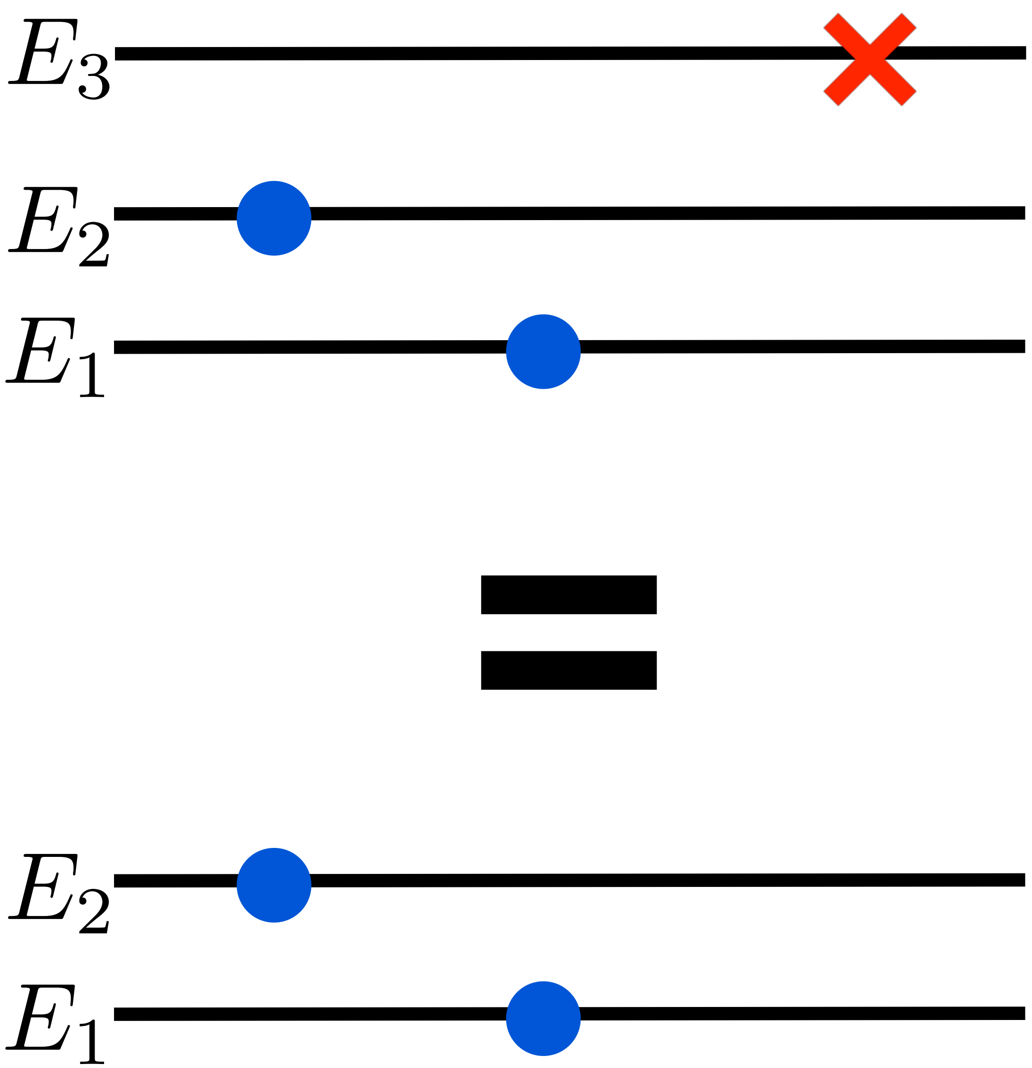

2-level system.

Impossible to have system in 3rd level
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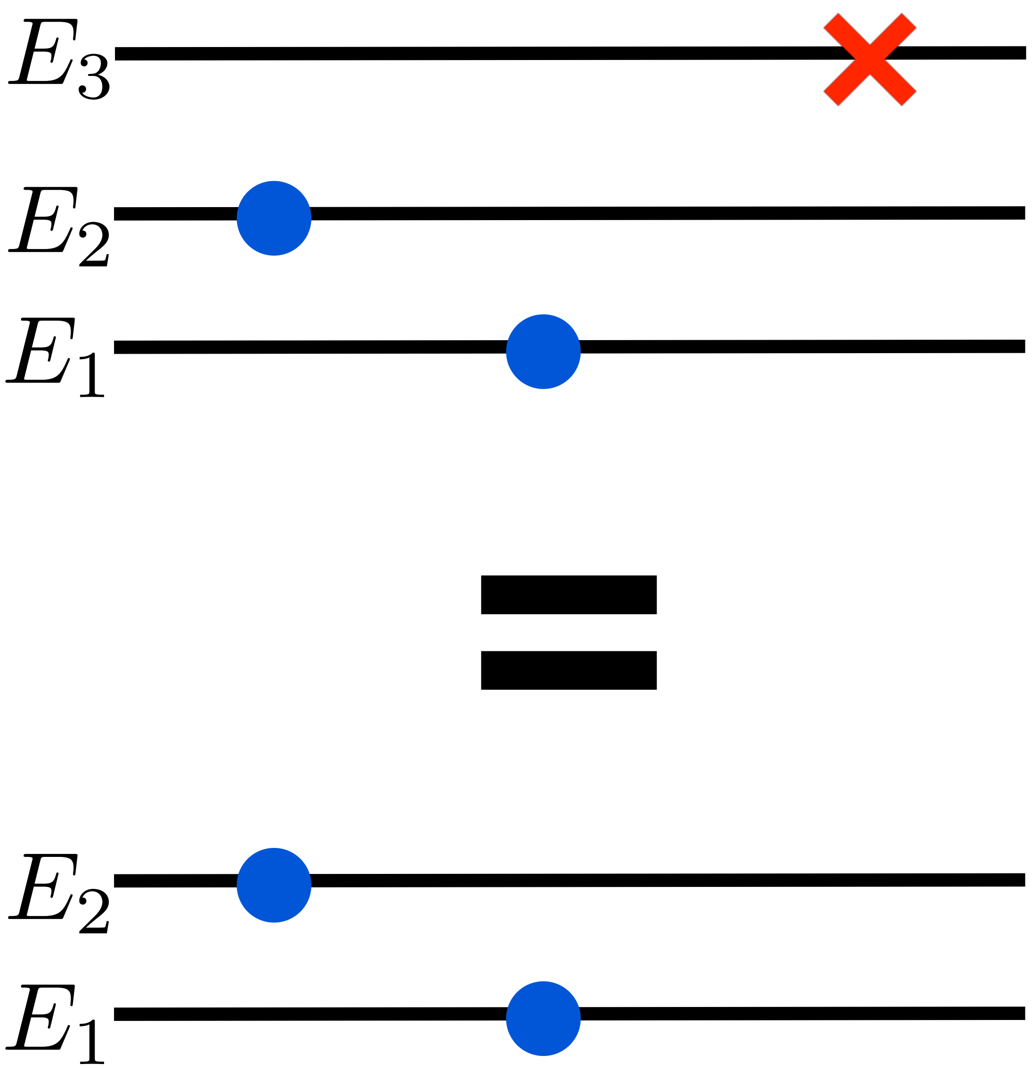

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QT: $\rho^{(3)} = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \rho^{(2)} = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$

CPT: $P^{(3)} = (P_1, P_2, 0) \longrightarrow P^{(2)} = (P_1, P_2)$

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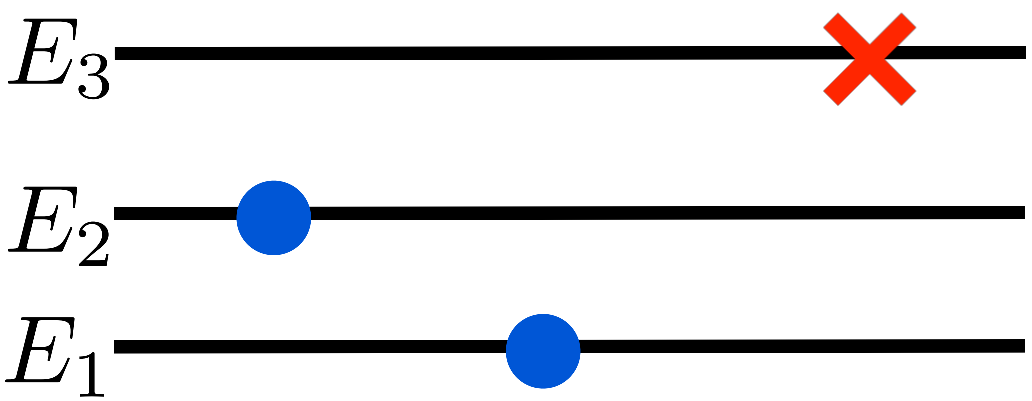

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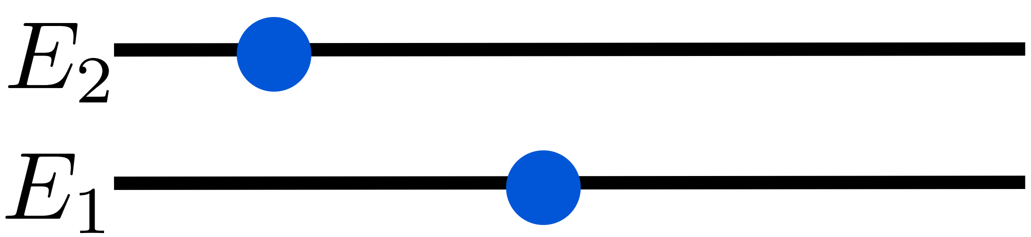

2-level system.

Otherwise, physics would be affected by impossible counterfactuals.

Axiom III: The Subspace Axiom

Axiom III: Let Ω_N and Ω_{N-1} be systems with capacities *N* and *N-1*. If $(E_1, \ldots, E_N)$ is a complete measurement on Ω_N, then the set of states ω with $E_N(\omega) = 0$ is equivalent to Ω_{N-1}.

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Capacity N of *Ω* = maximal # of perfectly distinguishable states.

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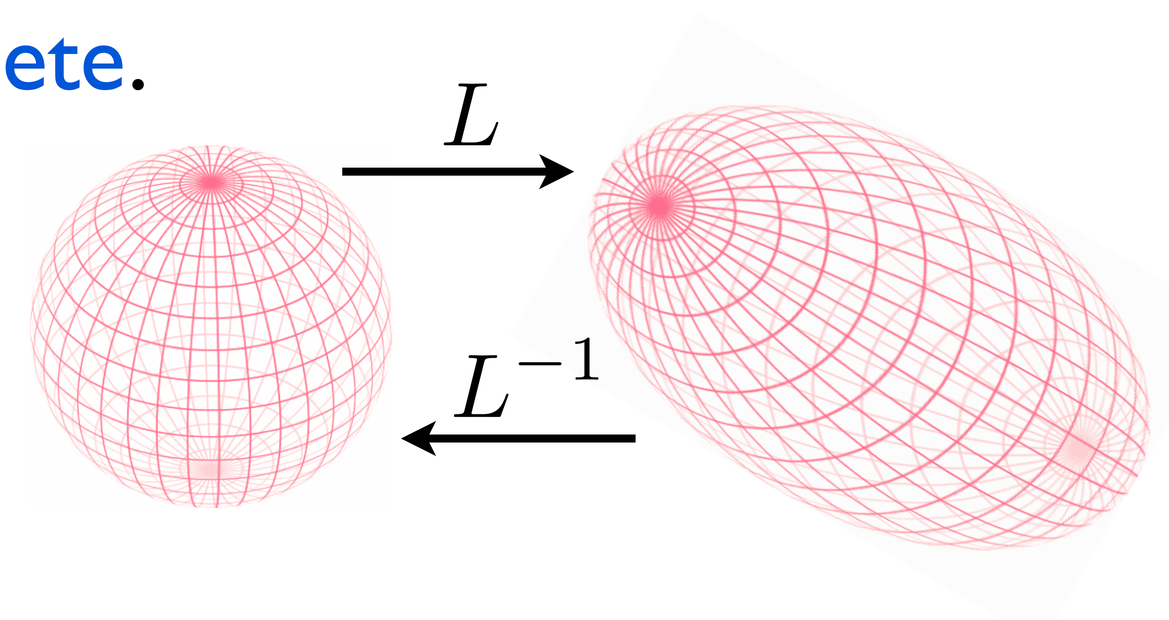
Equivalent = same state spaces up to a linear map (physically the same!)



Result

Ll. Masanes and M. P. Müller, *A derivation of quantum theory from physical requirements*, New J. Phys. **13**, 063001 (2011).

Theorem. There are exactly two GPTs satisfying the following three axioms:

I. Reversibility

II. Local Tomography

III. Subspace Axiom

These are:

- **Classical Probability Theory**

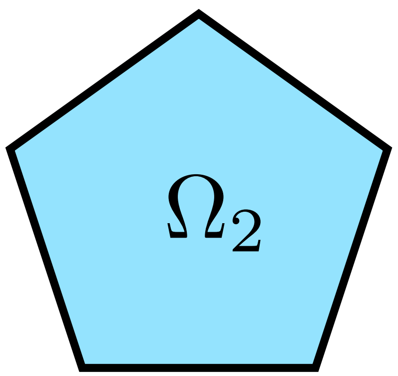
  Capacity-*n* states are the probability vectors $(p_1, \ldots, p_n)$;
  reversible transformations are permutations; the usual measurements.

- **Quantum Theory**

  Capacity-*n* states are the complex $n \times n$ density matrices;
  reversible transformations are unitary conjugations $\rho \mapsto U\rho U^\dagger$;
  the measurements are the POVMs.

Sketch of first small part of proof

Why a bit is described by a ball:



capacity 2 (bit)

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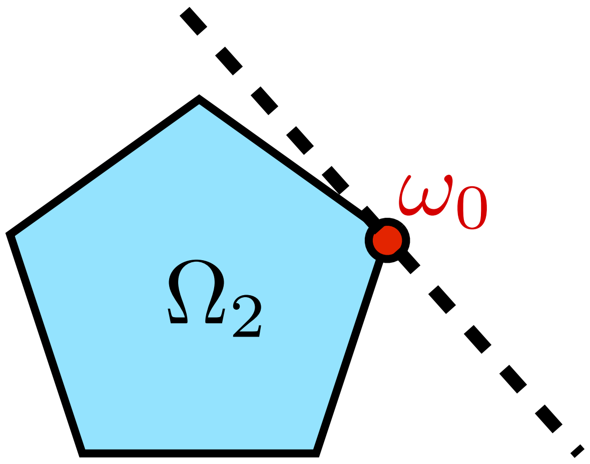

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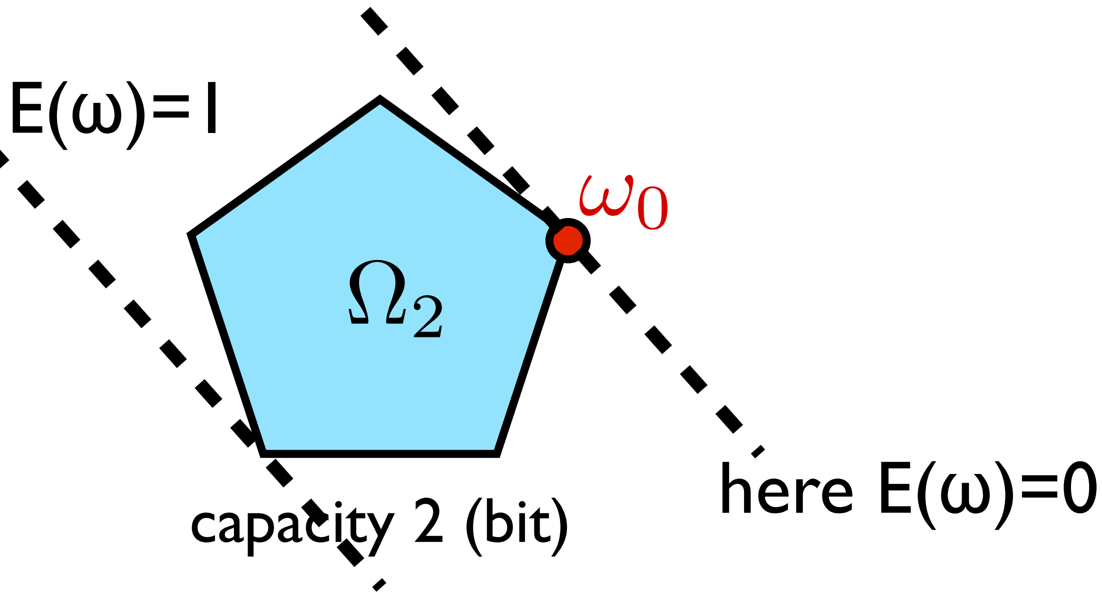

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$$\Rightarrow \{\omega : E(\omega) = 0\} = \{\omega_0\} \sim \Omega_1.$$

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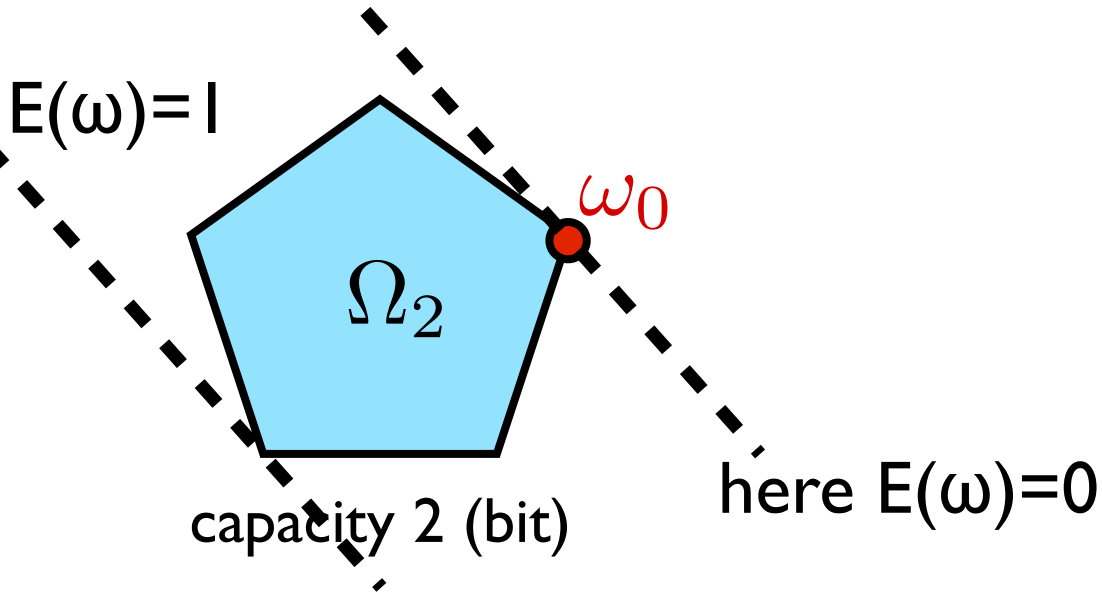

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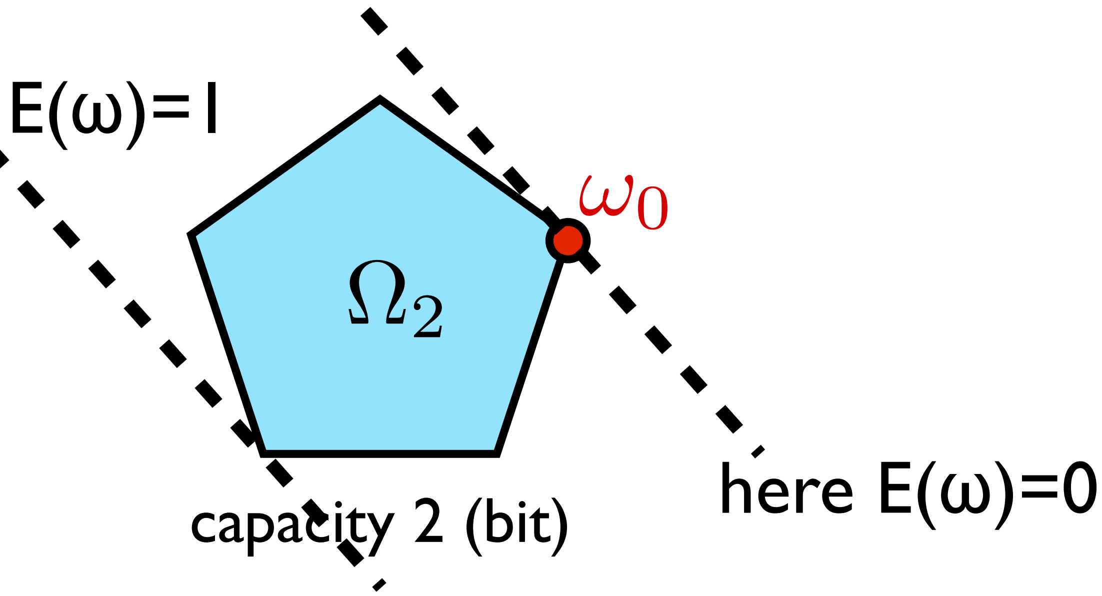

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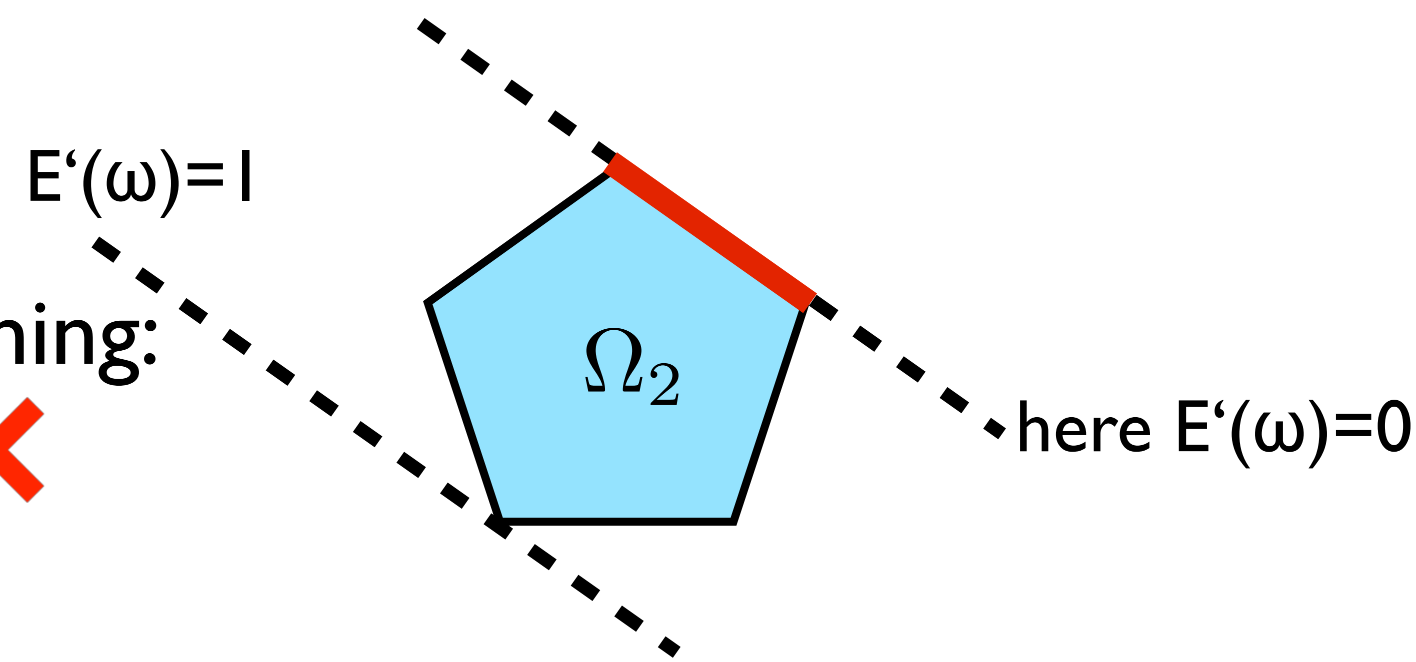

If there is a face, similar reasoning:
Ω_1 contains ∞ many states. ❌

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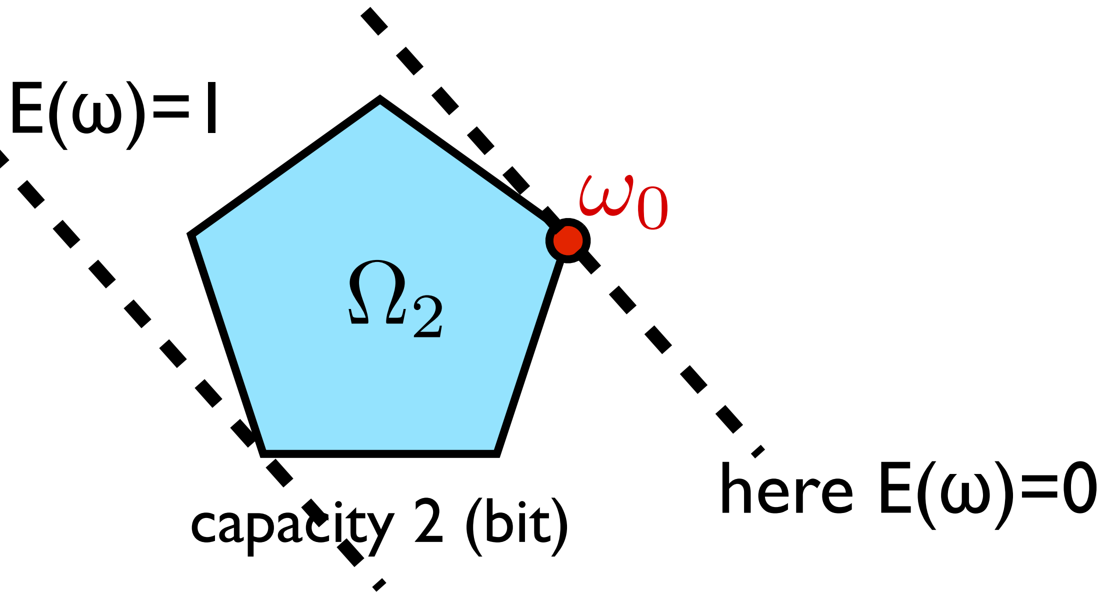

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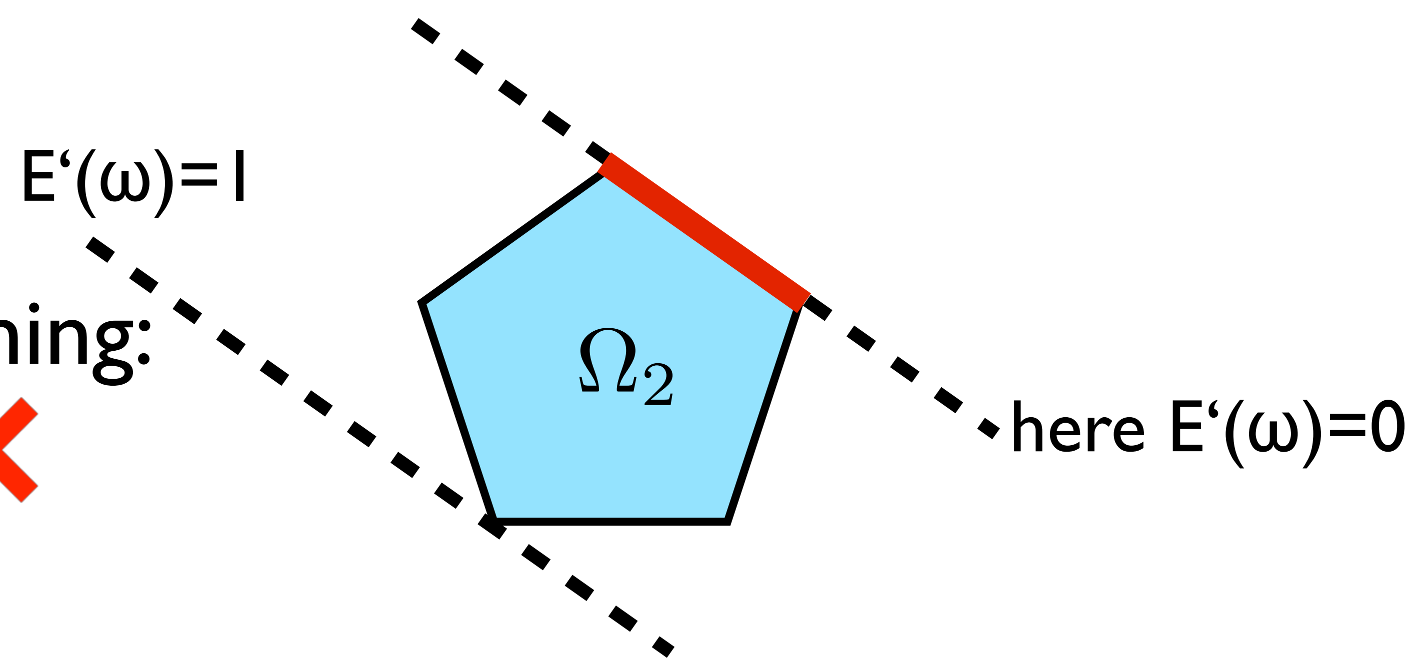

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⇒no faces: 

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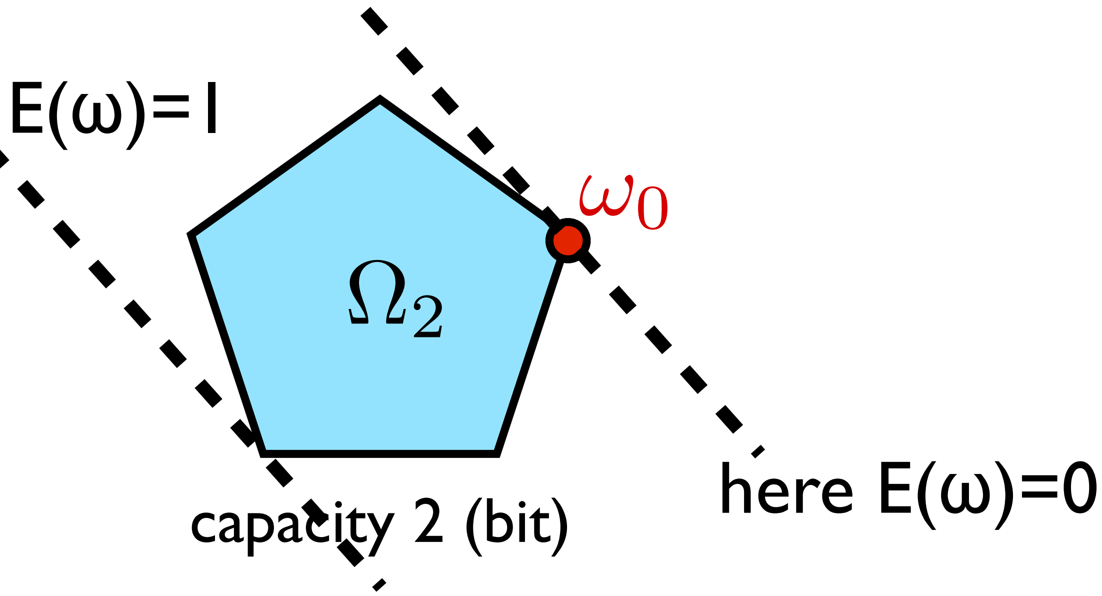

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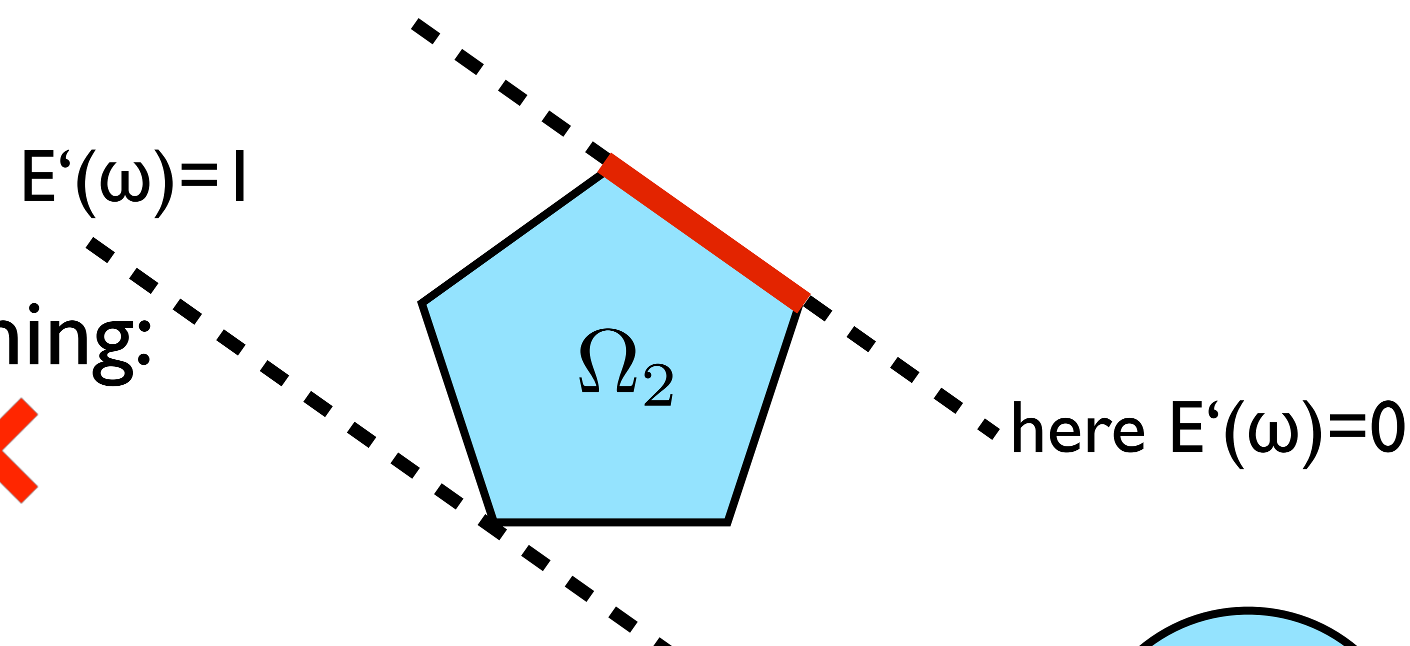

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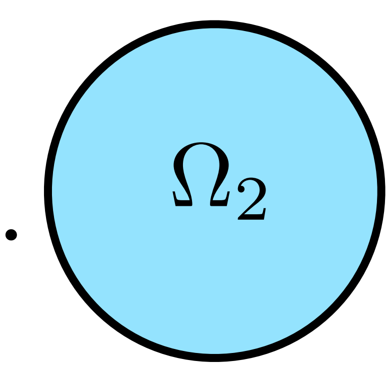
⇒no faces: 

Reversibility axiom $\Rightarrow$ Ω_2 is a ball. 

More on the blackboard:

- Formal definition of a **GPT system**,
- example: the "**gbit**" and its exploding bits model.