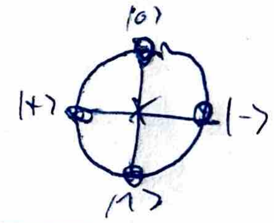
Lecture 3

Let us prove that qubit quant. theory is preparation-contextual.
Two prep. procedures satisfy $P \sim P'$ if and only if the resulting density matrices agree, $\rho_P = \rho_{P'}$.

Ex.: $P :=$ "prepare either $|0\rangle$ or $|1\rangle$ with prob. $\frac{1}{2}$"
$P' :=$ " " " $|+\rangle$ or $|-\rangle$ " " "
because $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}\mathbb{1}$

Operational equivalence in the operational theory
$\simeq$ convex-linear equation in the QT state space



$\boxed{\text{Prep.-noncontextuality of the qubit would mean: distribution } \mu(\lambda) \text{ depends only on } \rho,\ \mu = \mu_\rho.}$

Recall: $\mathrm{Prob}(k \mid P, M) = \int_\Lambda \underbrace{d\mu_P(\lambda)}_{\text{class. "state" (prob. distr.)}} \underbrace{\chi_{M,k}(\lambda)}_{\text{classical "effect"/ (response function)}}$

We use two facts:

Fact 1: [redacted] Two prob. distr. μ and μ' are classically perfectly distinguishable if and only if they don't overlap,
i.e. $\mu(\lambda)\mu'(\lambda) = 0$ for almost all λ.

(will ignore in the following)

[redacted] Fact 2: [redacted]

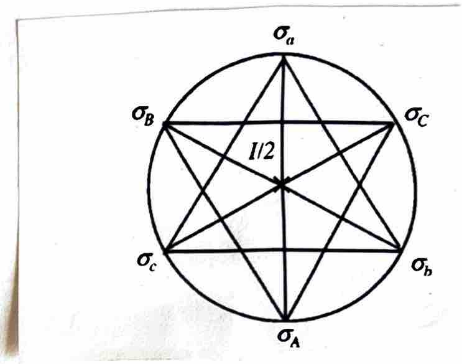
[redacted]

Convex mixtures of prep. procedures (and hence of states) are represented on the hidden-variable space as the corresponding convex mixtures of distributions. Discuss!

Theorem. The qubit does not admit of a noncontextual ontological model.

Proof (Spekkens 2005).

Consider the state vectors



$\psi_a = \begin{pmatrix}1\\0\end{pmatrix}$, $\psi_A = \begin{pmatrix}0\\1\end{pmatrix}$, $\psi_b = \begin{pmatrix}1/2\\ \sqrt{3}/2\end{pmatrix}$,

$\psi_B = \begin{pmatrix}\sqrt{3}/2\\ -1/2\end{pmatrix}$, $\psi_c =$ [redacted] $\begin{pmatrix}1/2\\ -\sqrt{3}/2\end{pmatrix}$ $\psi_C = \begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}$, and

the corresponding density matrices $\rho_a, \rho_A, \rho_b, \rho_B, \rho_c, \rho_C$.

ρ_a and ρ_A are perf. distinguishable because $\langle\psi_a|\psi_A\rangle = 0$, hence by Fact 1 [redacted]

(*) $\mu_a(\lambda)\mu_A(\lambda) = 0 \quad \forall \lambda \in \Lambda$.

Similarly $\mu_b(\lambda)\mu_B(\lambda) = 0$, $\mu_c(\lambda)\mu_C(\lambda) = 0 \quad \forall \lambda \in \Lambda$.

Furthermore, we have the operational equivalences

$\frac{1}{2}\mathbb{1} = \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A = \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B = \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C$

$= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c = \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C$.

[redacted] Assuming preparation-noncontextuality, this implies

$$\mu_{\frac{1}{2}\mathbb{1}}(\lambda) = \tfrac{1}{2}\mu_a(\lambda) + \tfrac{1}{2}\mu_A(\lambda)$$
$$= \tfrac{1}{2}\mu_b(\lambda) + \tfrac{1}{2}\mu_B(\lambda)$$
$$= \tfrac{1}{2}\mu_c(\lambda) + \tfrac{1}{2}\mu_C(\lambda)$$
$$= \tfrac{1}{3}\mu_a(\lambda) + \tfrac{1}{3}\mu_b(\lambda) + \tfrac{1}{3}\mu_c(\lambda) \qquad (**)$$
$$= \tfrac{1}{3}\mu_A(\lambda) + \tfrac{1}{3}\mu_B(\lambda) + \tfrac{1}{3}\mu_C(\lambda)$$

Fix some $\lambda \in \Lambda$.
Now, (*) implies that one of the pair $\mu_a(\lambda)$ and $\mu_A(\lambda)$ must be zero; similarly for the other pairs. We can check all those (8) possibilities; let's only discuss

$$\mu_a(\lambda) = \mu_b(\lambda) = \mu_c(\lambda) = 0.$$

$$(**) \Rightarrow \mu_{\frac{1}{2}\mathbb{1}}(\lambda) = 0$$

The other equations above imply $\mu_A(\lambda) = \mu_B(\lambda) = \mu_C(\lambda) = 0$, i.e. the all-zero solution.
Since this is true for all λ, we obtain $\mu_{\frac{1}{2}\mathbb{1}} \equiv 0$, which is a contradiction.
$\square$

The proof above already suggests a structural reinterpretation of Spekkens' noncontextuality:

$$\underset{\text{state}}{S} \longmapsto \underset{\text{distribution}}{\mu_S} \qquad \underbrace{\lambda S + (1-\lambda)S' \longmapsto \lambda\mu_S + (1-\lambda)\mu_{S'}}_{\text{linearity}}$$

and similarly for the effects (i.e. measurement-outcome pairs).

D. Schmid et al., Characterization of Noncontextuality in the Framework of Generalized Probabilistic Theories, PRX Quantum 2, 010331 (2021).

<u>Recall</u>:

<u>Operational theory</u> with prep. procedures $\{P\}$, measurement outcome-pairs $\{[M,k]\}$

(form equivalence classes)

<u>GPT system</u> (A, Ω_A, E_A)

every $\omega \in \Omega_A$ is an equivalence class $[P]$,
every $e \in E_A$ is an equivalence class $[M,k]$ [redacted]

<u>Theorem</u> (Schmid et al.)
Given some operational theory, there exists a noncontextual ontological model (of finite cardinality) if and only if the associated GPT system is simplex-embeddable.

(will ignore this)

"Simplex-embeddable" $\simeq$ linearly embeddable into a classical GPT

<u>Def</u>: A GPT (A, Ω_A, E_A) is simplex-embeddable if there is some dimension $d \in \mathbb{N}$ [redacted] and two linear maps $i: A \to \mathbb{R}^d$ and $\kappa: A^* \to (\mathbb{R}^d)^* \simeq \mathbb{R}^d$ such that $i(\Omega_A) \subseteq \Delta_d$ and $\kappa(E_A) \subseteq \Delta_d^*$, such that $(e_A, \omega_A) = (\kappa(e_A), i(\omega_A))$ $\forall \omega_A \in \Omega_A, e_A \in E_A$.

Here $\Delta_d = \left\{ \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix} \,\middle|\, 0 \le p_i \le 1 \ \forall i,\ \sum_i p_i = 1 \right\}$ (classical states),

$$\Delta_d^* = \left\{ \begin{pmatrix} e_1 \\ \vdots \\ e_d \end{pmatrix} \,\middle|\, 0 \le e_i \le 1 \ \forall i \right\} \text{ (classical effects).}$$

<u>Example 1</u>: Spekkens' proof shows that the qubit is not simplex-embeddable.

<u>Example 2</u>: The "stabilizer qubit" <u>is</u> simplex-embeddable and, in this sense, noncontextual. **STOP!**

Single-qubit stabilizer subtheory ~~(see e.g. PRL 122, 140405 (2019))~~:

Allow only preparations (and measurements) in the eigenbases of the single-qubit Pauli matrices $\{X, Y, Z\}$, and convex combinations of those.

This has a noncontextual ontological model with ~~$\Lambda = \{\pm 1\}^3$~~; ~~$\lambda = (x, y, z)$ denote the eigenvalues of the Pauli operators, Preparing the +1 eigenstate of X corresponds to setting x=+1 and choosing y and z uniformly at random, etc.~~ $|\Lambda| = 4$, i.e. a classical 4-level system. Picture: see slides

This <u>seems</u> to hint at some relation

contextuality $\Longleftrightarrow$ quantum-computational speedup,

because of the <u>Gottesman-Knill theorem</u>:

> n-qubit stabilizer states, Clifford unitaries + Pauli measurements can be perfectly simulated in polynomial time on a classical computer.

Indeed, M. Howard et al., Contextuality supplies the 'magic' for quantum computation, Nature 510, 351–355
makes a strong claim along these lines. But it's subtle.
What they show: For odd-prime-dim qudits,

negativity of the discrete Wigner function

$\Longleftrightarrow$ state-dependent Kochen–Specker contextuality wrt. stabilizer meas.

[redacted]

$\Longleftarrow$ useful for universal magic-state computation.

Furthermore, if the operational theory has a full d-dim quantum system as its GPT)

Spekkens' noncontextuality [redacted]

$\Rightarrow$ sharp measurements satisfy <u>outcome determinism</u>,

[redacted] i.e. $\chi_{M,k}(\lambda) \in \{0,1\}$ for almost all λ

$\Rightarrow$ Kochen–Specker noncontextuality.

Hence, in the above scenario,

useful for universal magic-state comp $\Rightarrow$ Spekkens' contextual

~~Recall the gbit and the exploding-bits model~~

~~Theorem: The gbit is not simplex-embeddable.~~

~~Proof: all 4 pure states of the gbit are pairwise perfectly dist.~~
~~A simplex embedding map i would map them to 4 pairwise perf. dist. classical distributions $\Rightarrow$ $i(\omega_j) \perp i(\omega_k)$ for $j \neq k$~~
~~$\Rightarrow$ $\dim i(\Omega_A) \geq 3$ for the gbits but $\dim \Omega_A = 2$ ↯.~~