ÖAW AUSTRIAN ACADEMY OF SCIENCES

IQOQI – INSTITUTE FOR QUANTUM OPTICS AND QUANTUM INFORMATION VIENNA

IQI

Generalized probabilistic theories and reconstructions of quantum theory

Markus P. Müller

Institute for Quantum Optics and Quantum Information (IQOQI), Vienna

Perimeter Institute for Theoretical Physics (PI), Waterloo, Canada

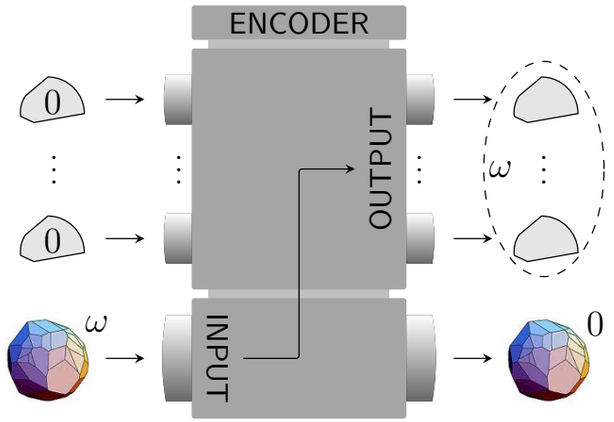


PI PERIMETER INSTITUTE

Overview

1. An (incomplete and physicists') historical perspective

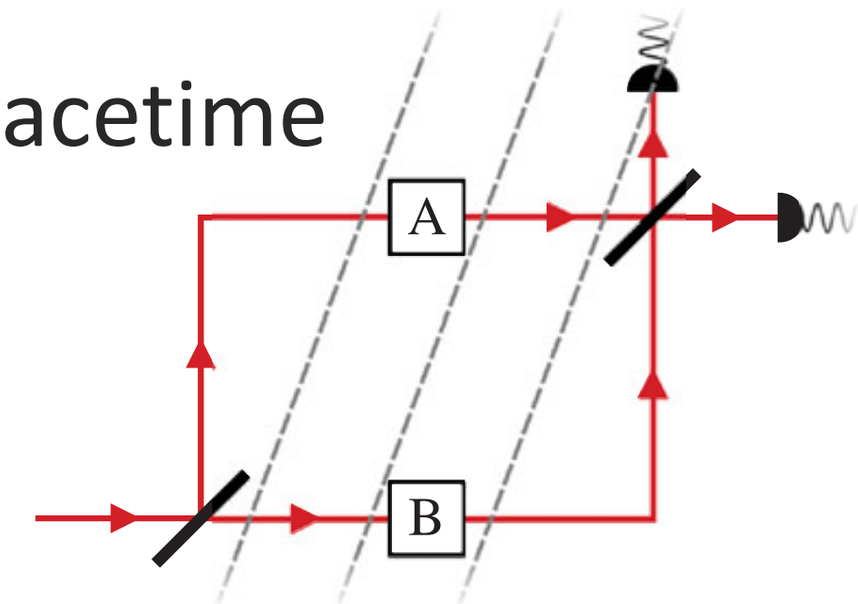
2. Generalized probabilistic theories



3. An example reconstruction of QT

4. Beyond reconstructions: QT and spacetime

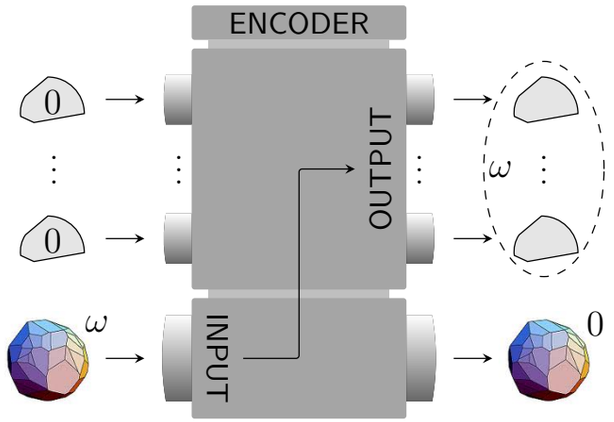
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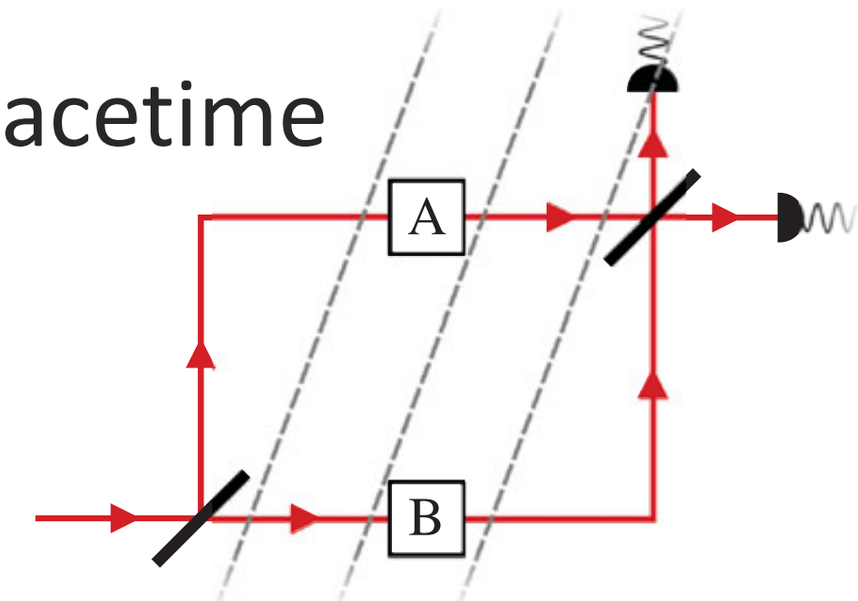
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An incomplete historical perspective

From quantum logic to quantum reconstructions

1936–2011 • Selected milestones in two interacting research traditions

QUANTUM LOGIC & EXPERIMENTAL PROPOSITIONS

1936



Garrett Birkhoff & John von Neumann
The Logic of Quantum Mechanics (1936)
Sharp questions → non-Boolean lattice of subspaces

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Gleason (1957): probability measures on projections → density operators

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George Mackey
1963
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J.-M. Jauch & C. Piron
Properties → generalized
Hilbert space (Piron, 1964)
Jauch: Foundations (1968) • *Piron: Foundations (1976)*

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Information–disturbance; Bell correlations; no-signalling security

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Finite-dimensional GPTs; continuous reversibility distinguishes quantum from classical

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2011: A WAVE OF OPERATIONAL RECONSTRUCTIONS

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TIME
compressed

→ development within a research tradition
- - → conceptual connection

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References and portrait credits: quantum-reconstruction-sources.md

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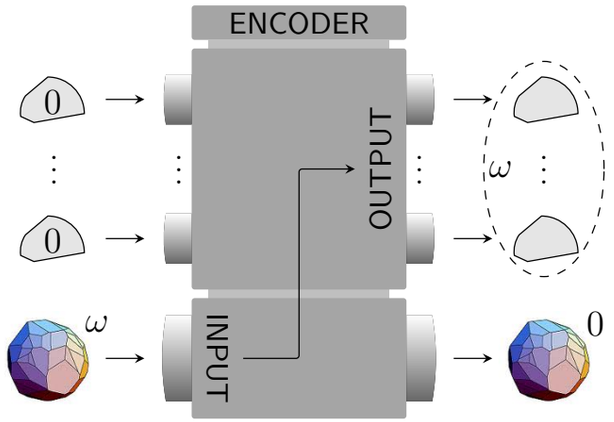
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From about 2000 on: impact of **quantum information theory** (focus on finite-dim. systems, compositional structure).

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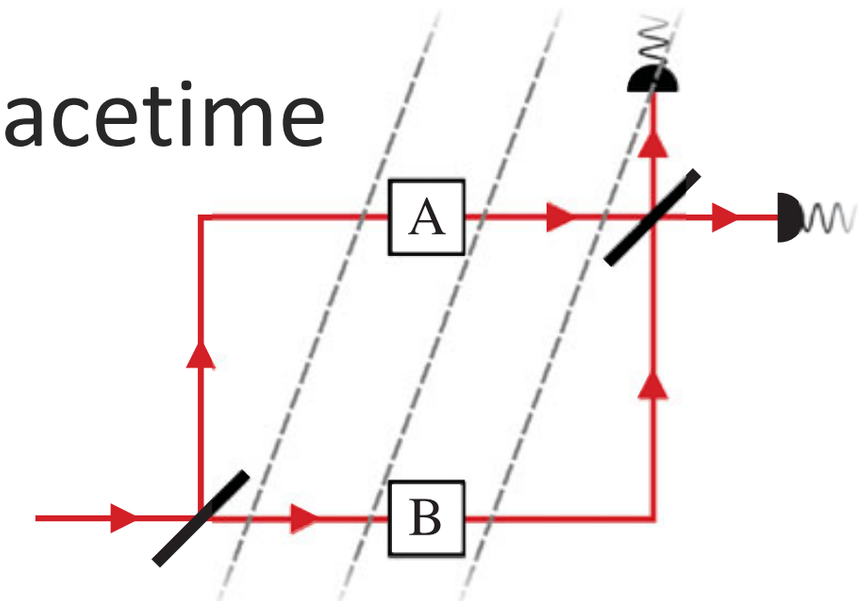
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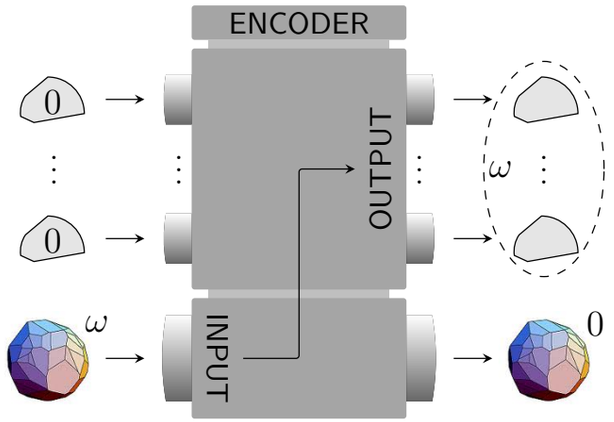


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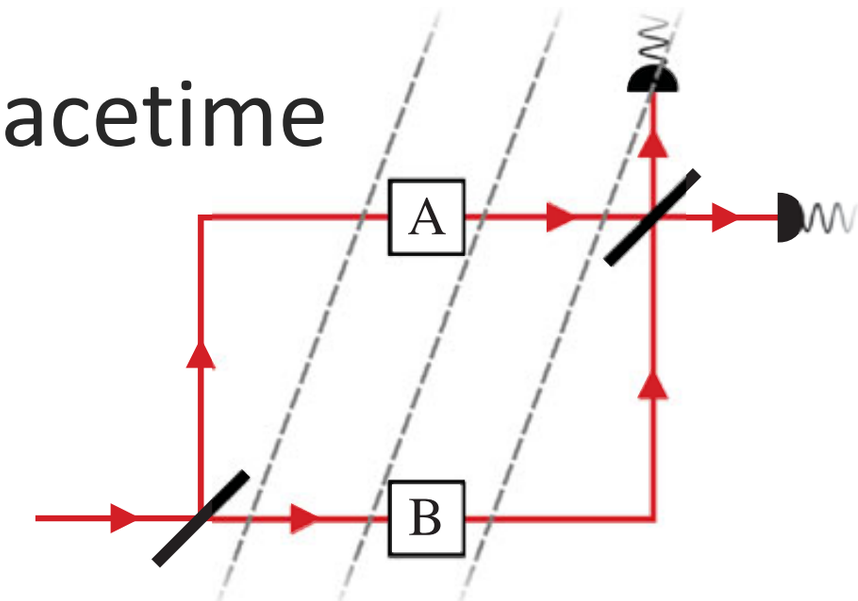
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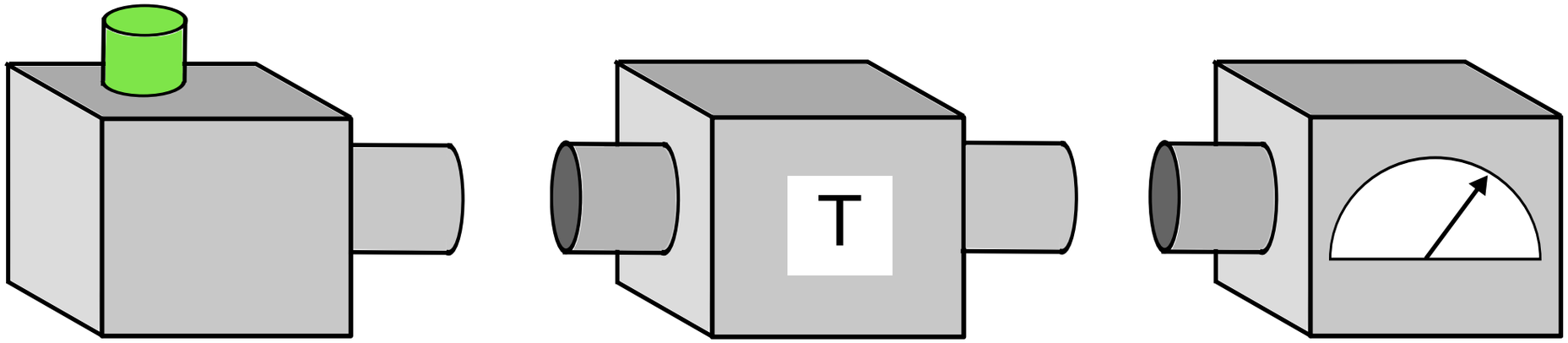


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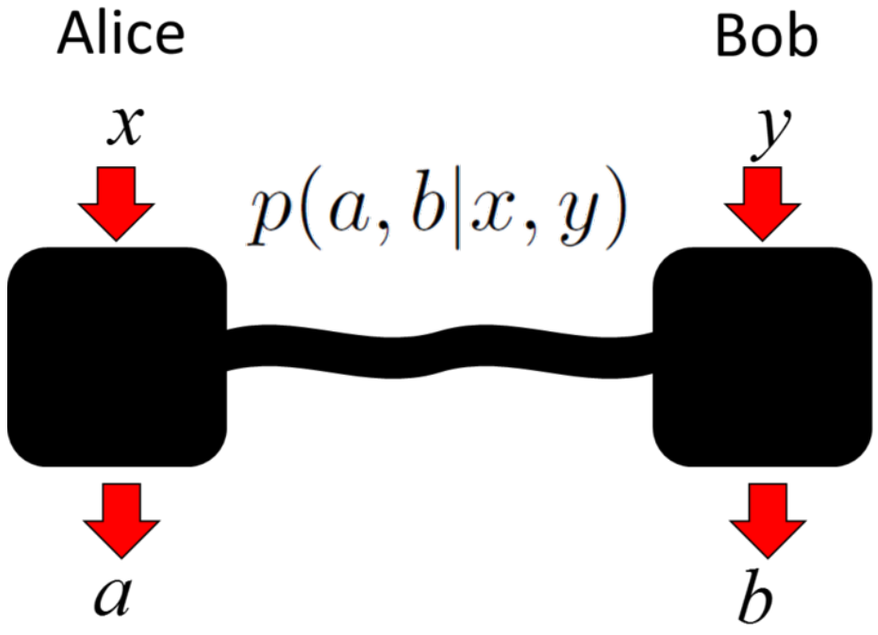
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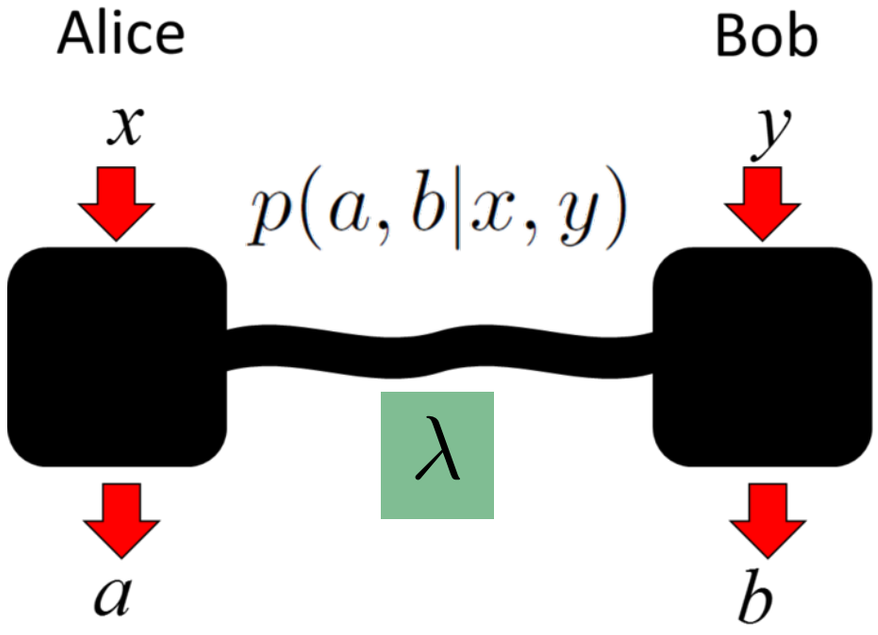
- For the same reason, the framework is **agnostic to the interpretation of probability**. A Bayesian interpretation is arguably always possible. What is implicitly often understood in the GPT context: the idea that we can repeat a prescribed experiment many times and subject probability claims to **statistical testing** via observed frequencies. This does not imply a frequentist *interpretation* of probability.

More general than quantum?

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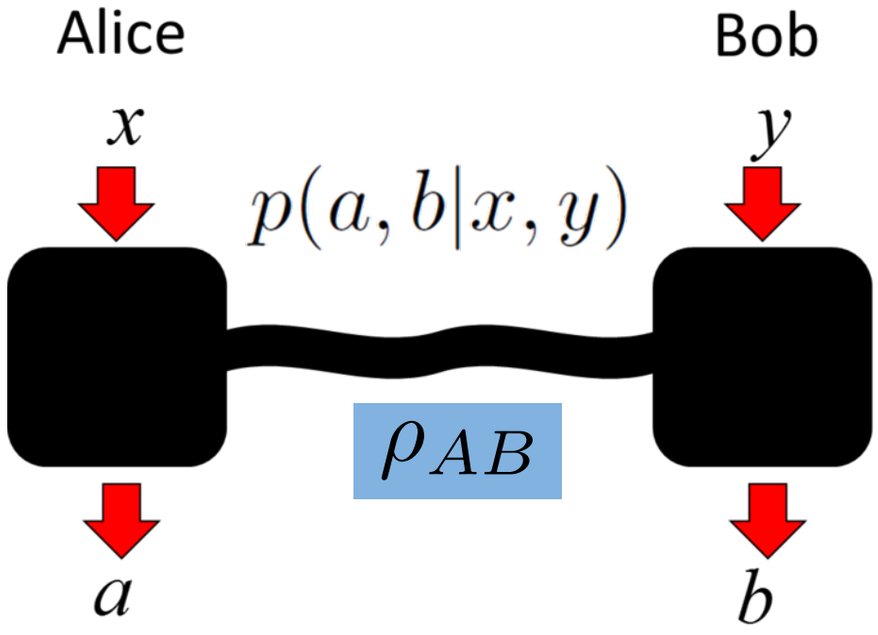
More general than quantum?



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$$P(a,b|x,y) = \sum_{\lambda\in\Lambda} P_A(a|x,\lambda)P_B(b|y,\lambda)P_\Lambda(\lambda)$$

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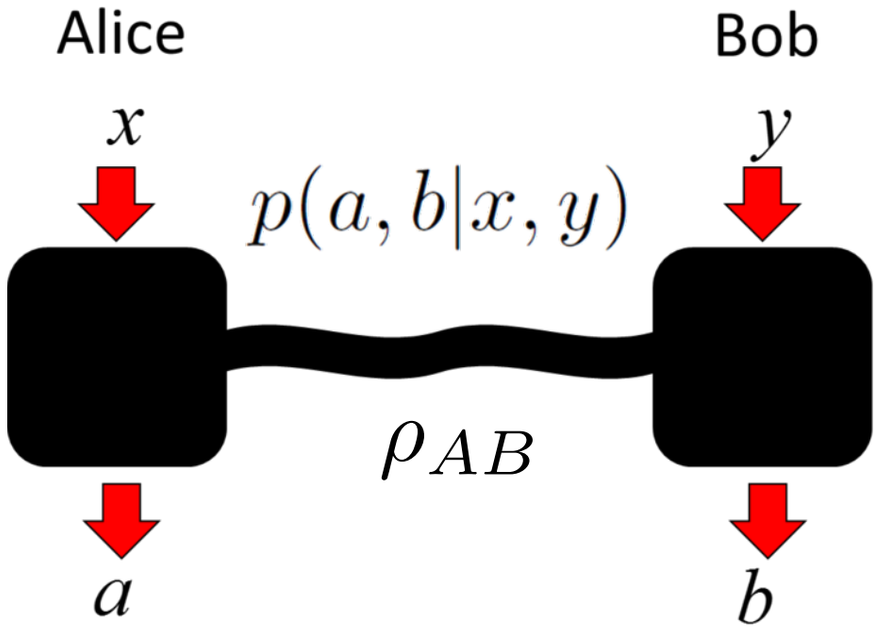
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No-signalling conditions:

$P(a|x,y)$ is independent of y,

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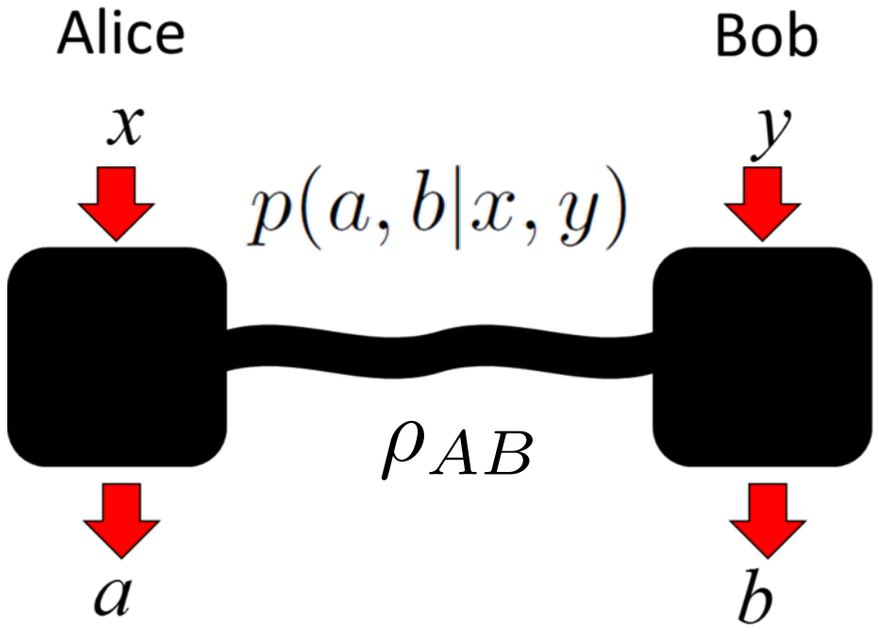
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Quantum admits more general *P*'s due to the **violation of Bell inequalities**.

The Bell-CHSH inequality

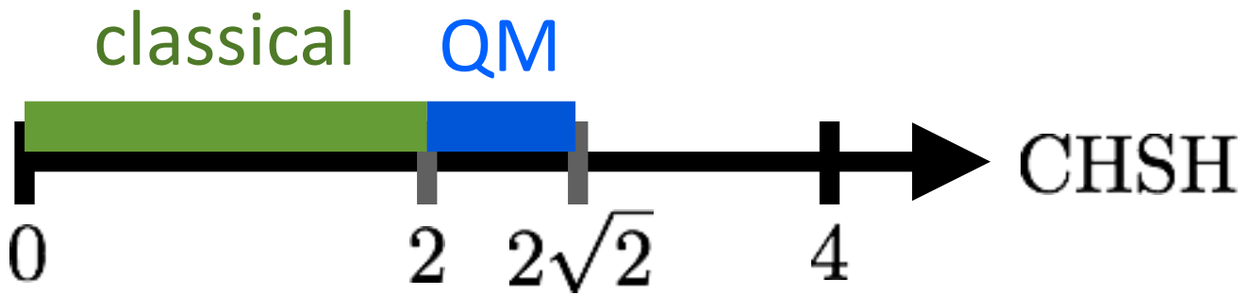
Classical probability distributions satisfy Bell inequality:

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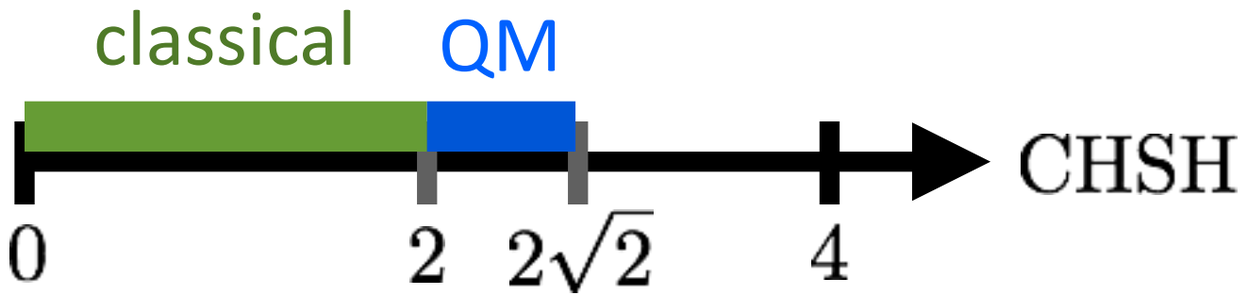
Quantum: Bell inequality violation.

$$\mathrm{CHSH} \leq 2\sqrt{2}.$$

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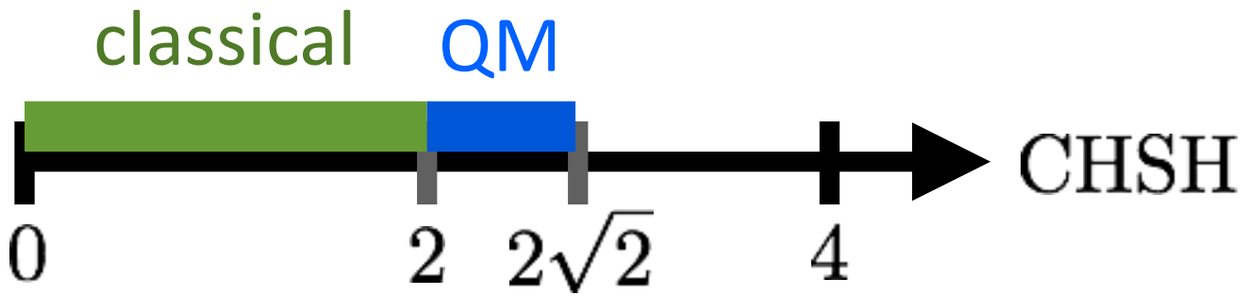
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No! Counterexample: the PR-box correlations

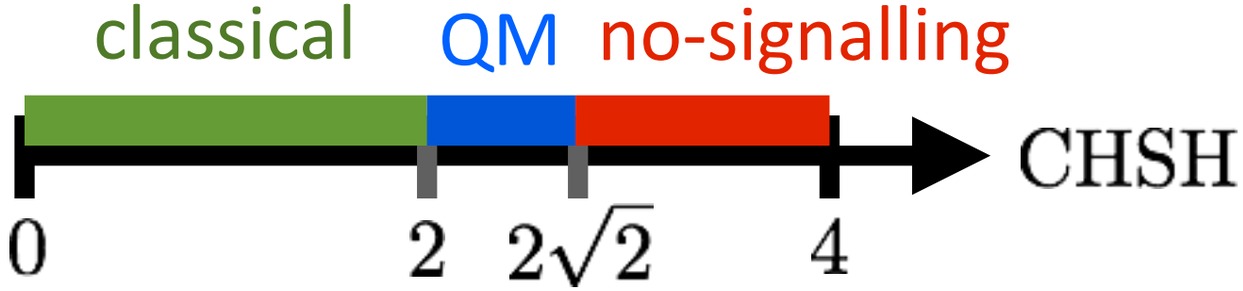
$$\begin{aligned} P(+1,+1|x,y) &= P(-1,-1|x,y) = \frac{1}{2} \\ &\text{if } (x,y) \in \{(0,0),(0,1),(1,0)\} \\ P(+1,-1|1,1) &= P(-1,+1|1,1) = \frac{1}{2} \end{aligned}$$

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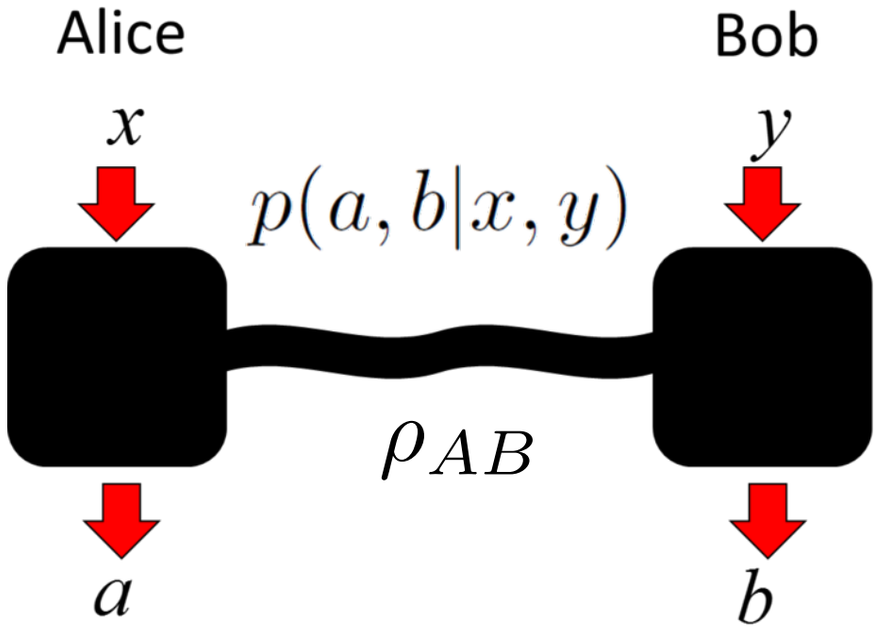
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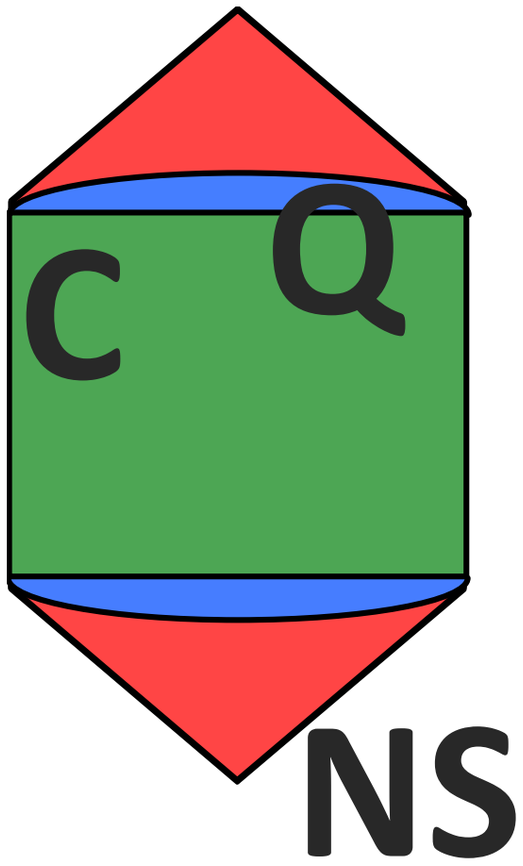
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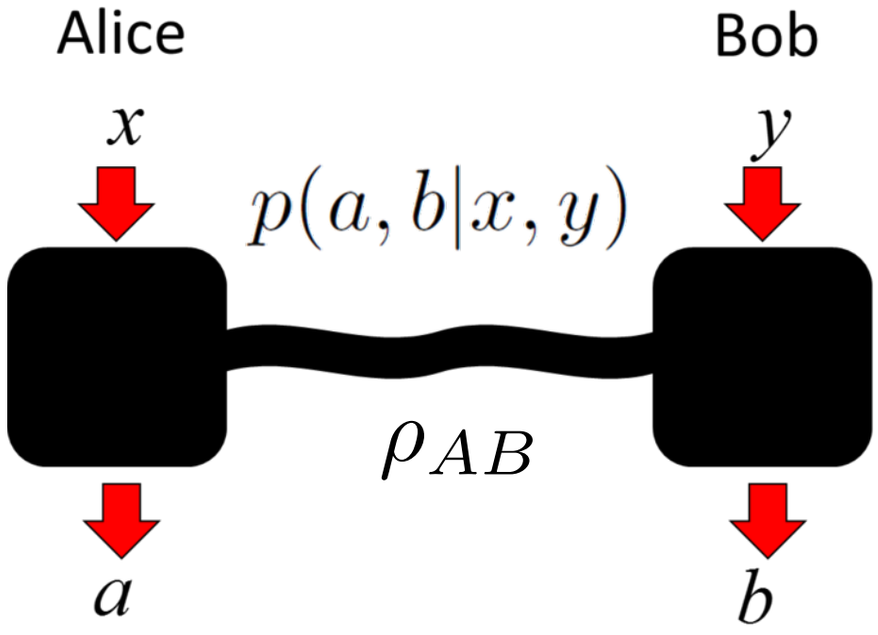
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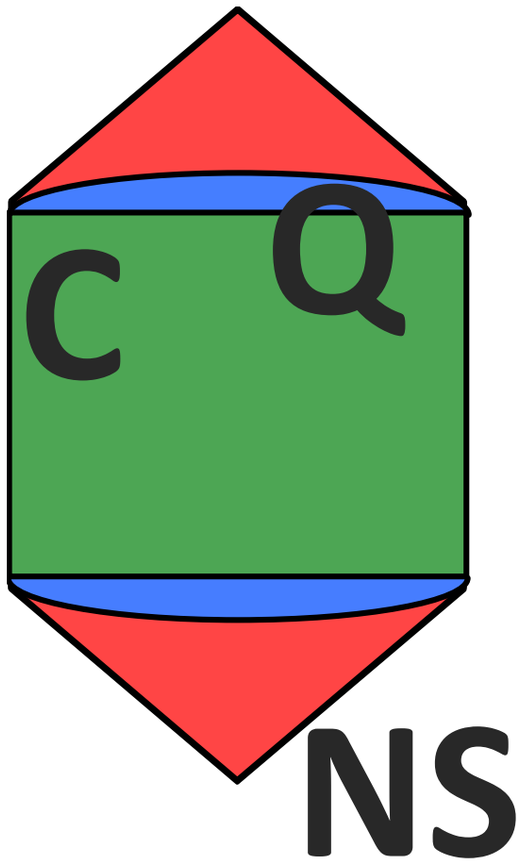
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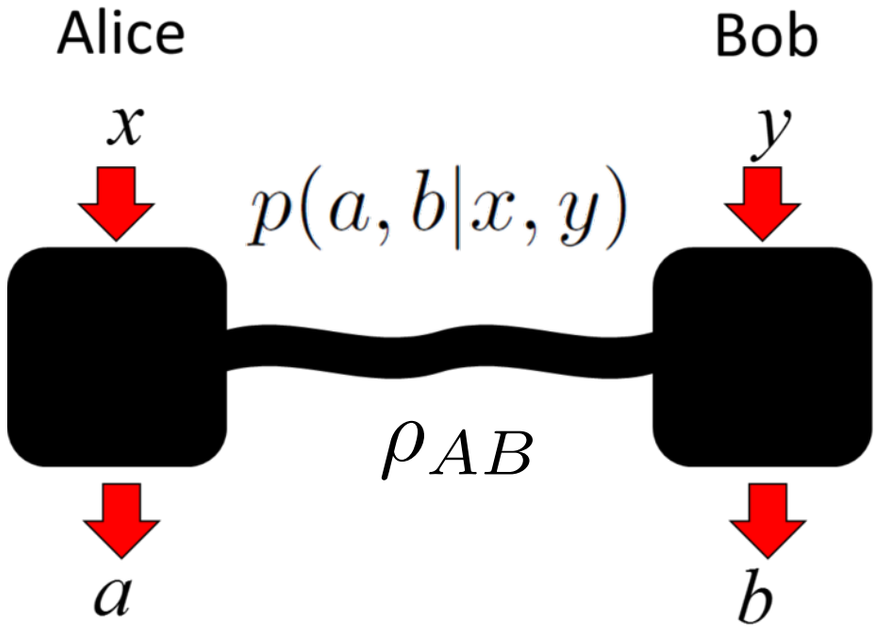
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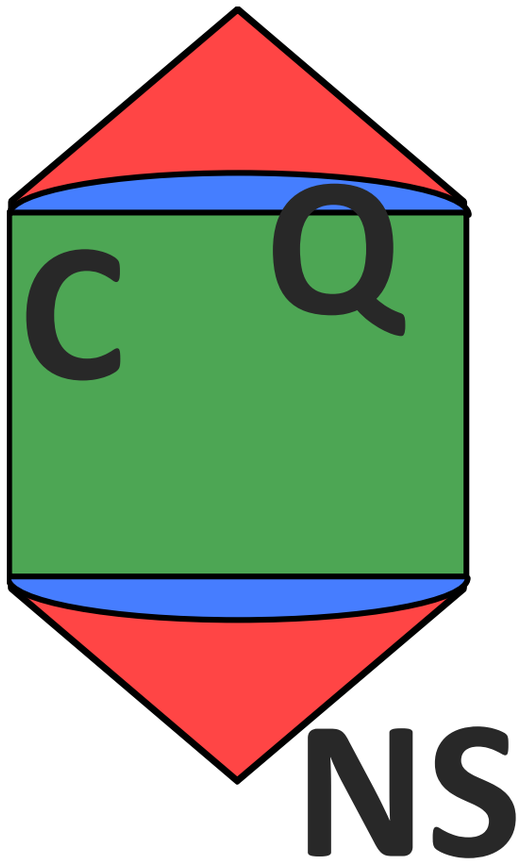
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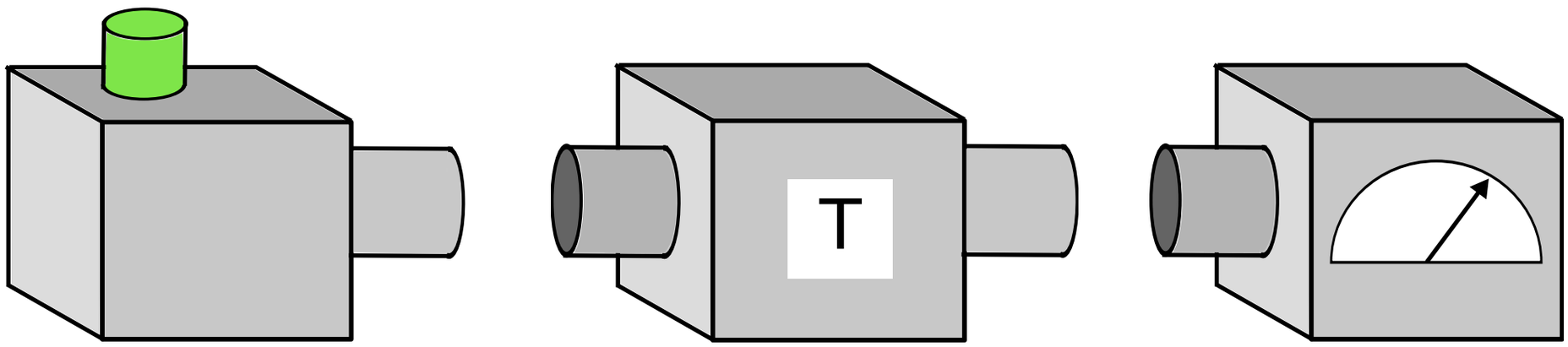
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3 examples of a "**generalized probabilistic theory**".

Generalized probabilistic theories



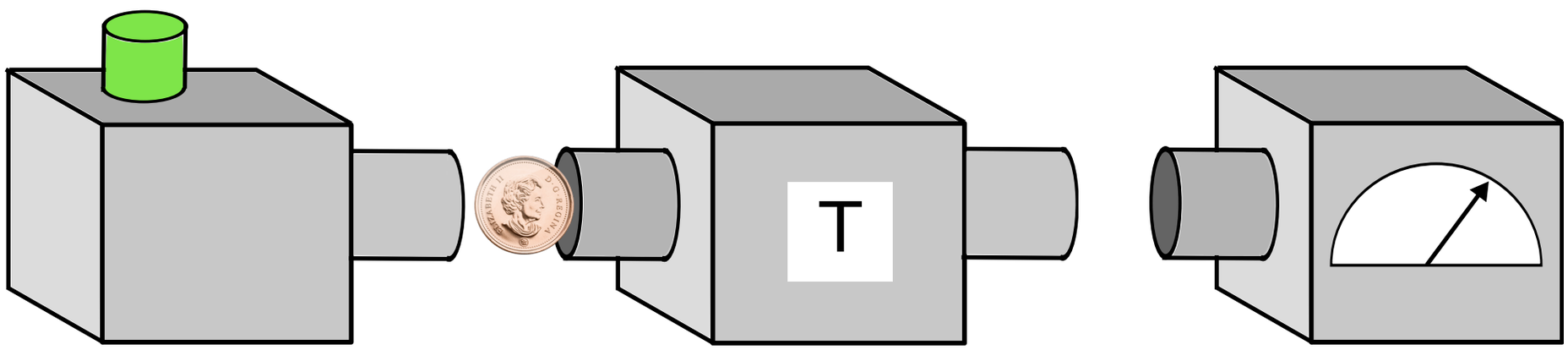
Preparation transformation measurement

Generalized probabilistic theories

Example: classical coin toss.



- On every push of button, the preparation device performs a biased coin toss.

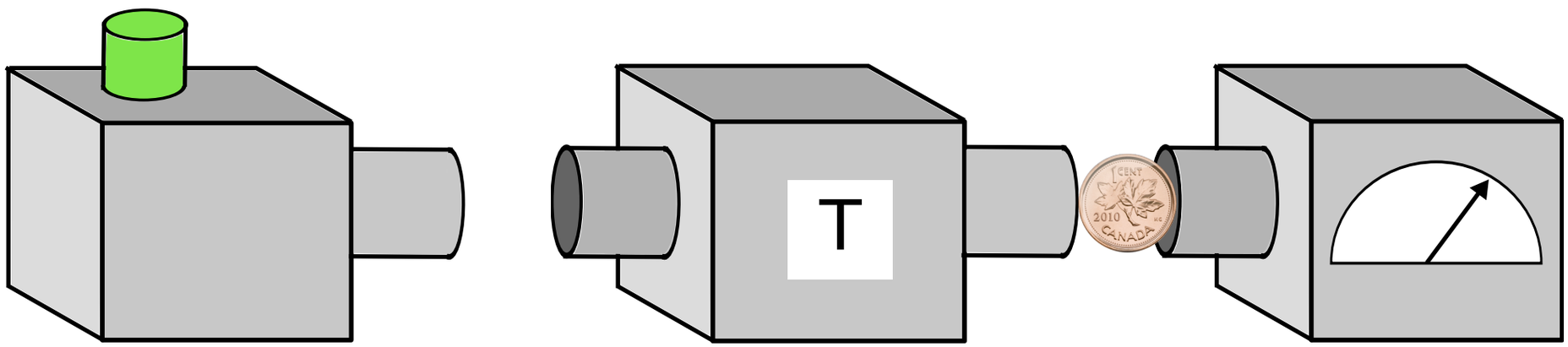


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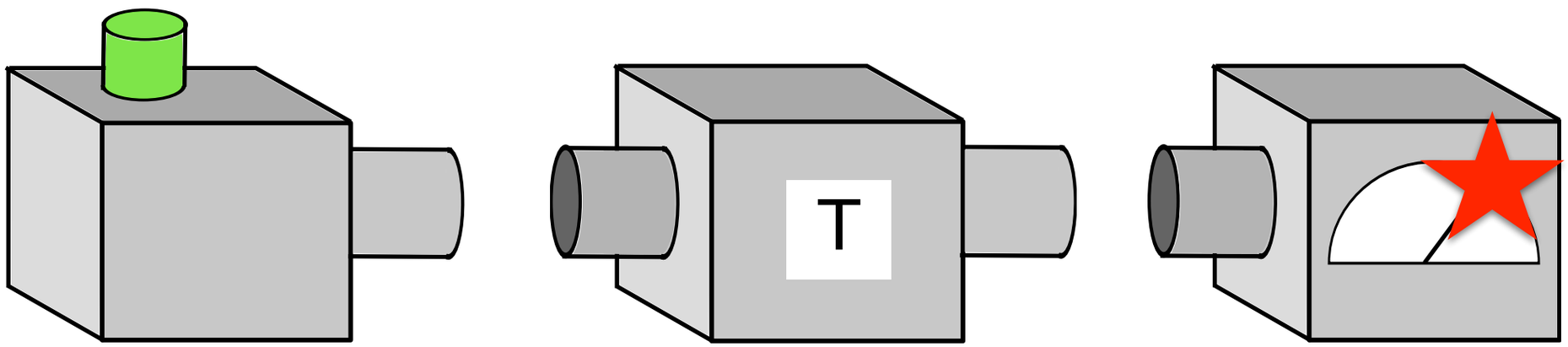


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- The measurement outcome is "heads" or "tails".



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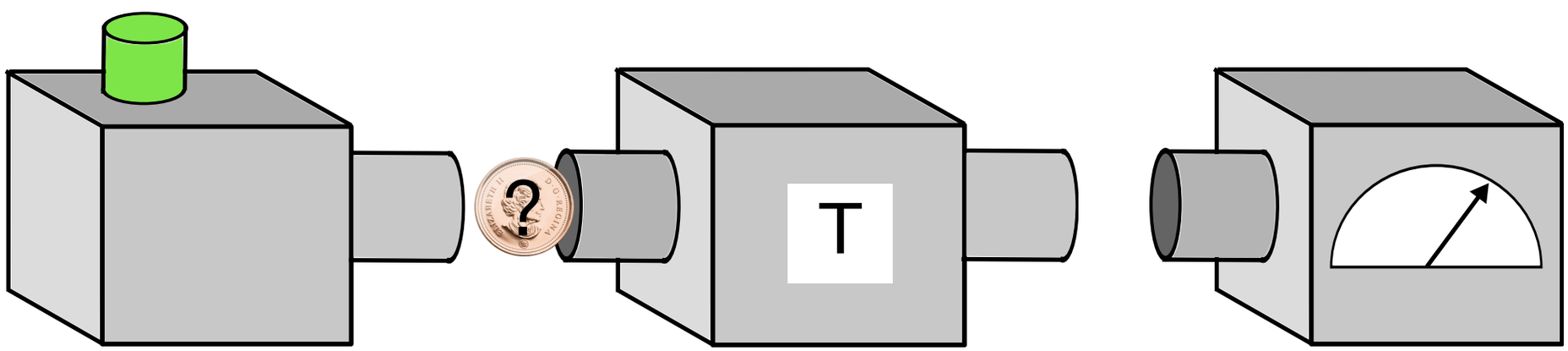
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- The preparation device prepares a physical system in a state ω. Here

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Generalized probabilistic theories

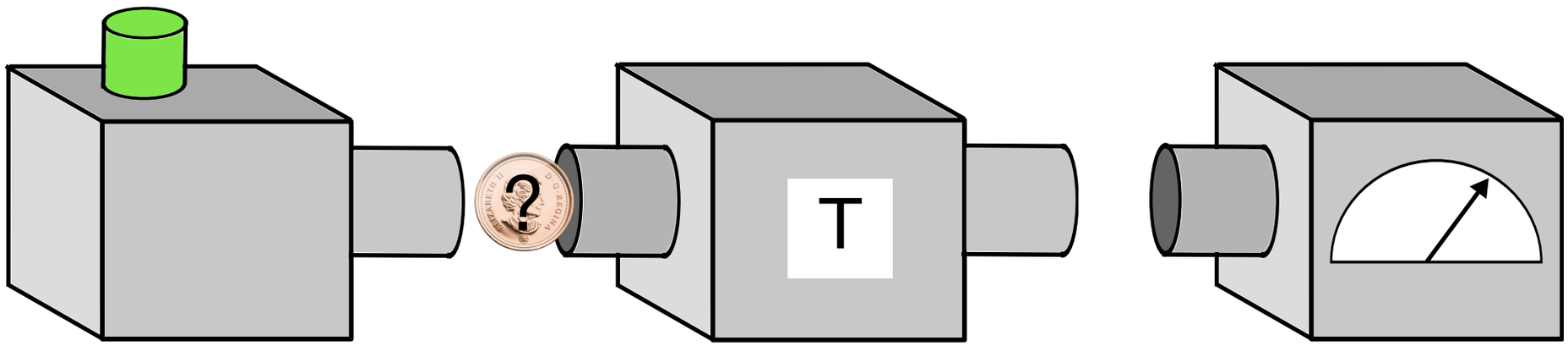
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State space Ω: the set of all possible states



Preparation transformation measurement

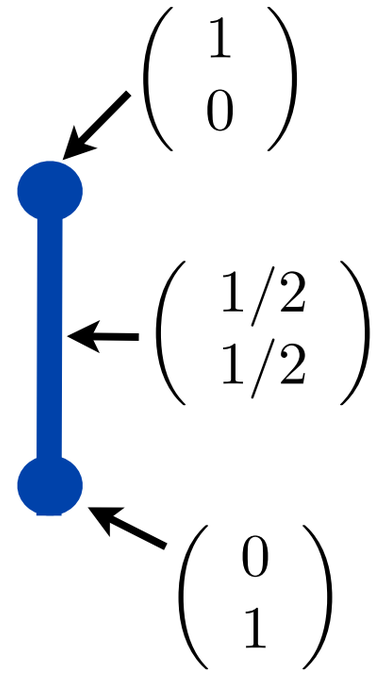
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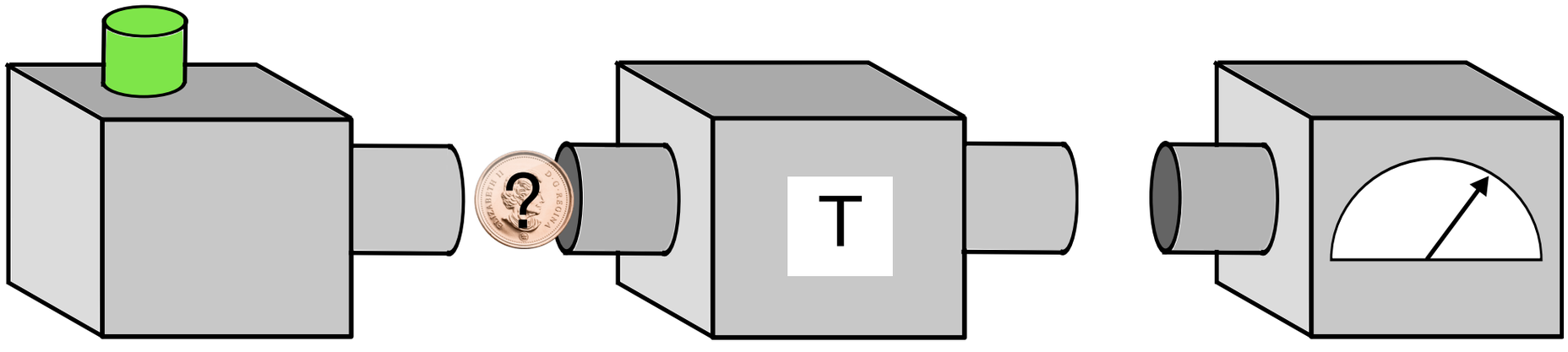


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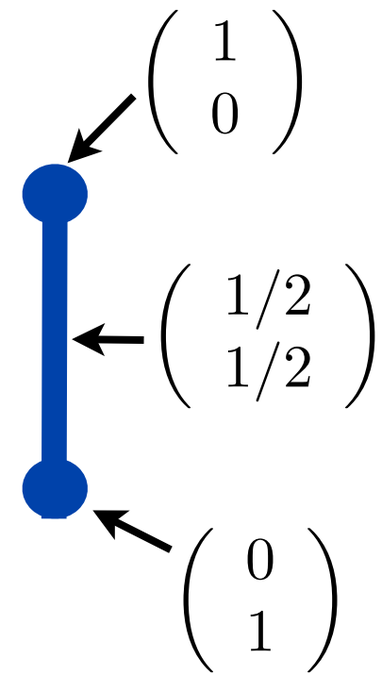
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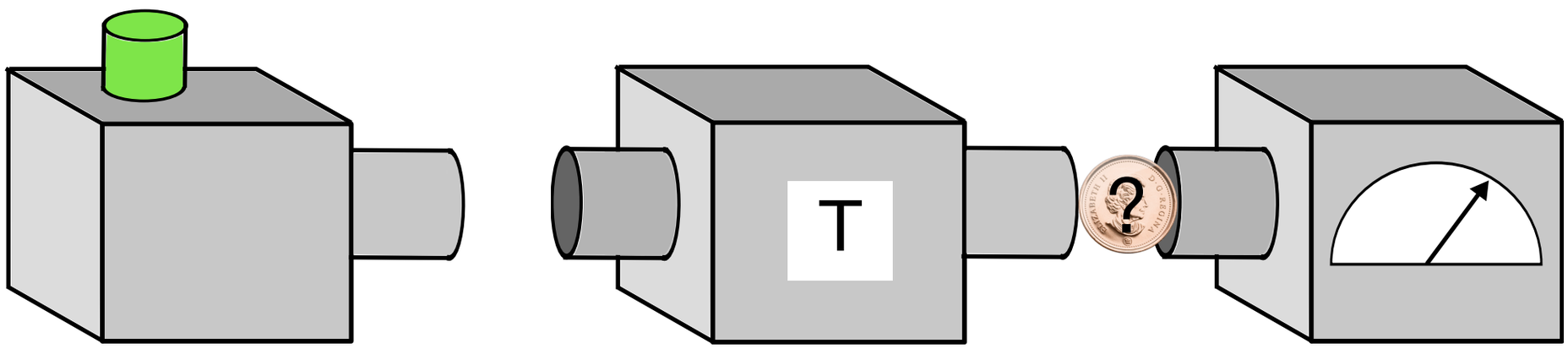
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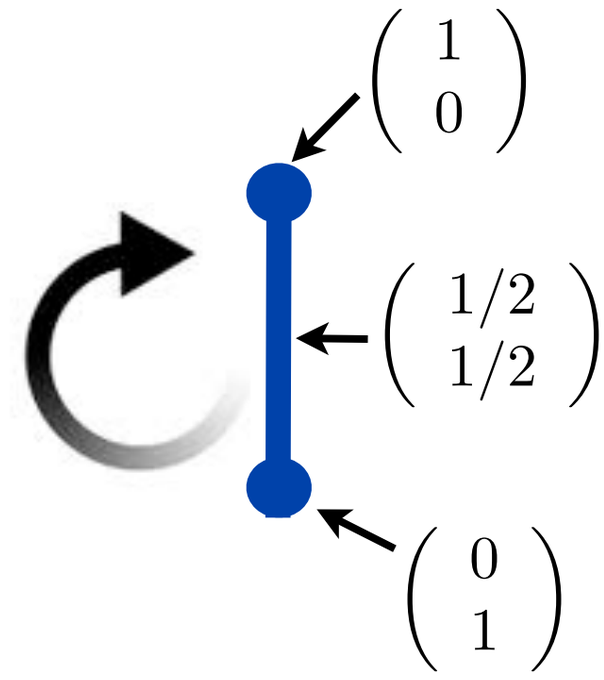
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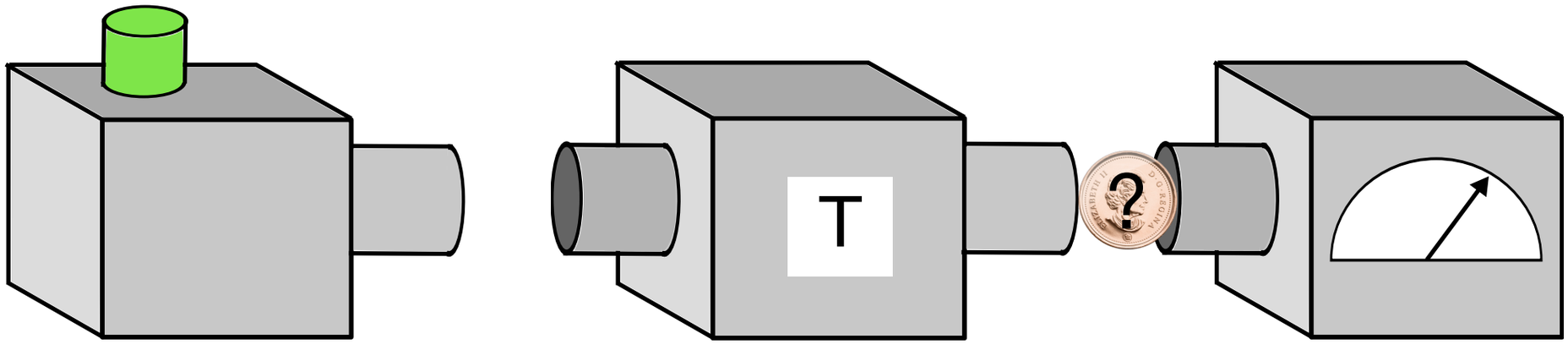
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Maps states to states and is linear.



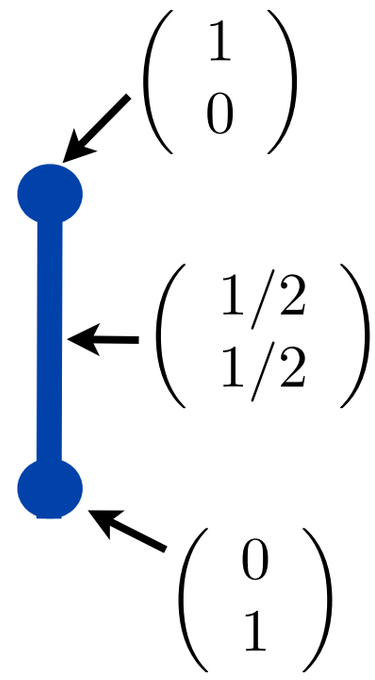
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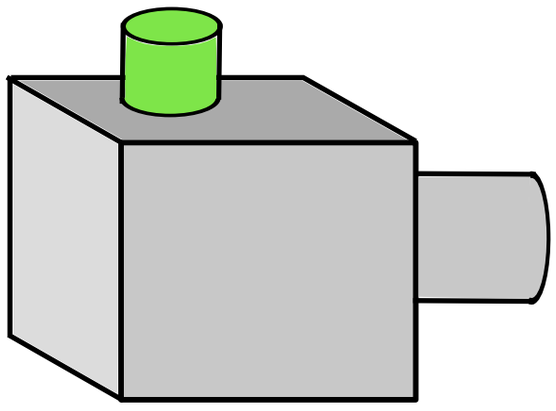
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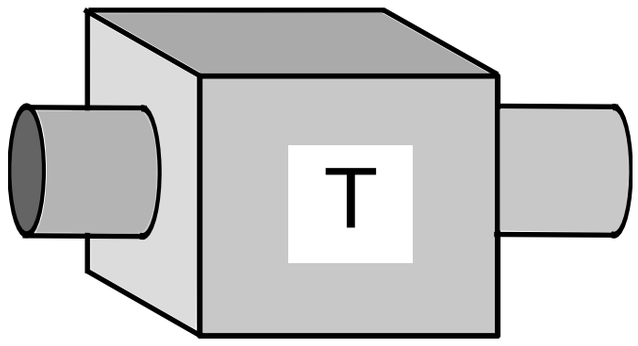


- Every measurement outcome has a probability, depending linearly on the state:

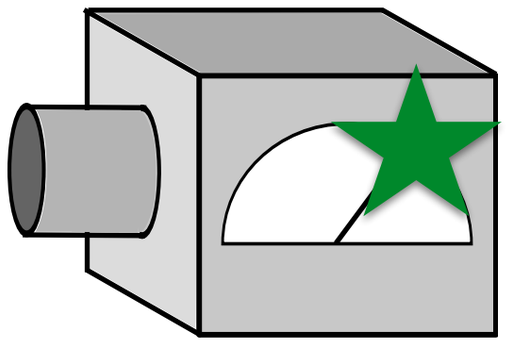




Preparation



transformation



measurement

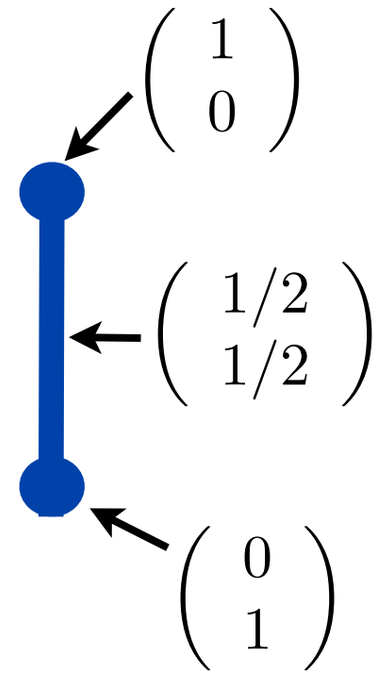
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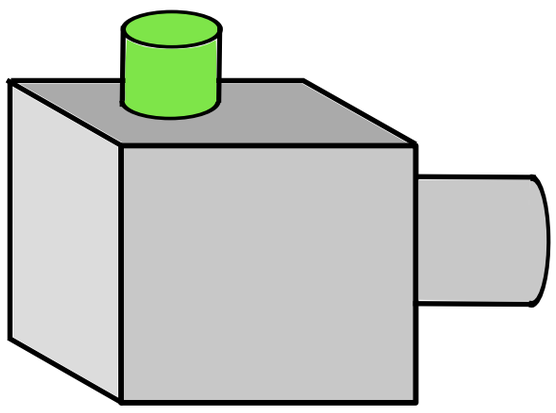
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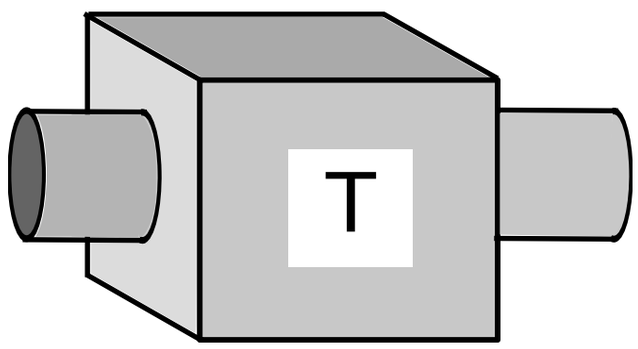
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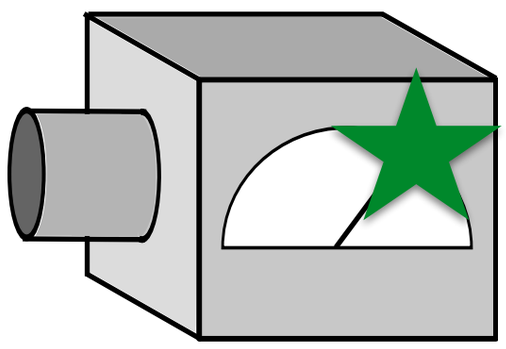




Preparation



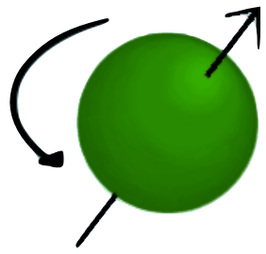
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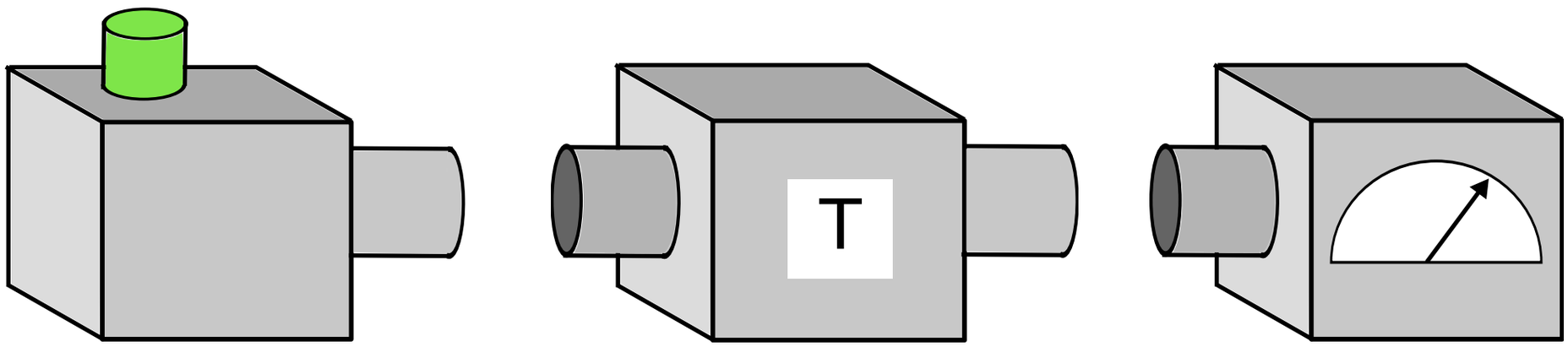


measurement

Generalized probabilistic theories

Example: quantum spin-1/2 particle.

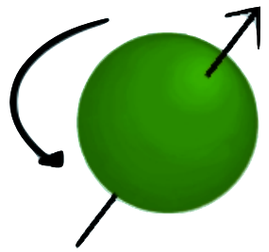




Preparation transformation measurement

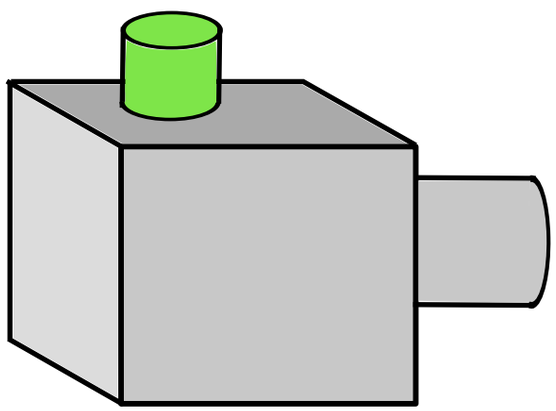
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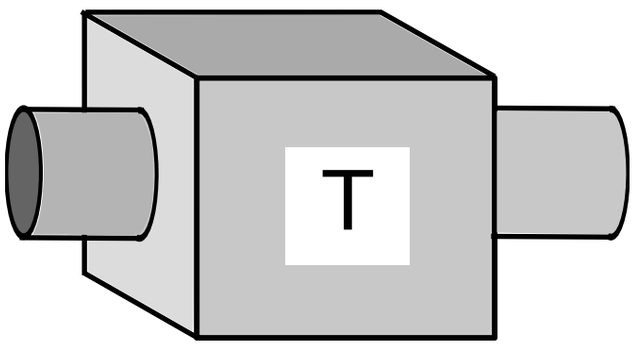


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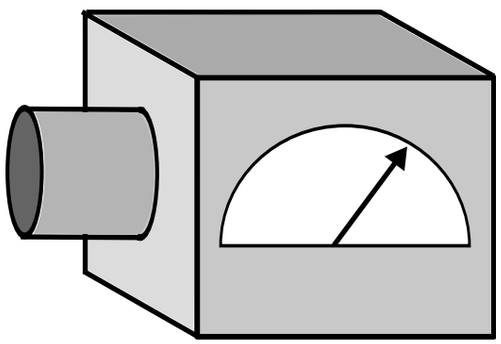
$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$$



Preparation



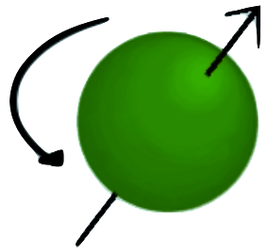
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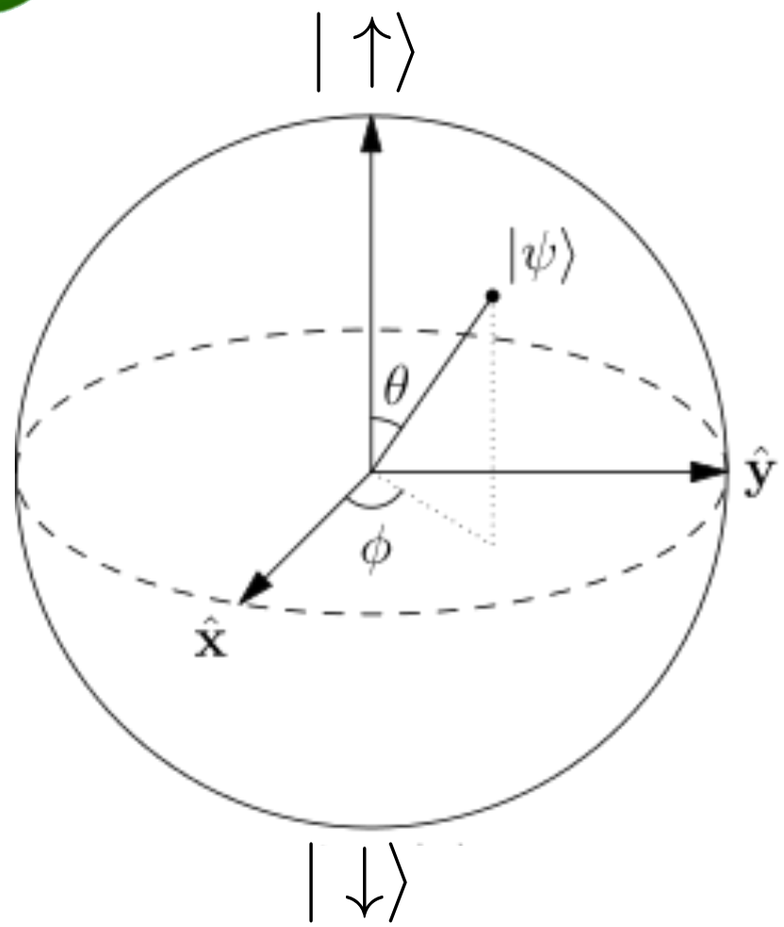


measurement

Generalized probabilistic theories

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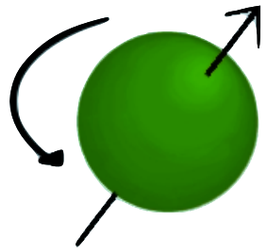


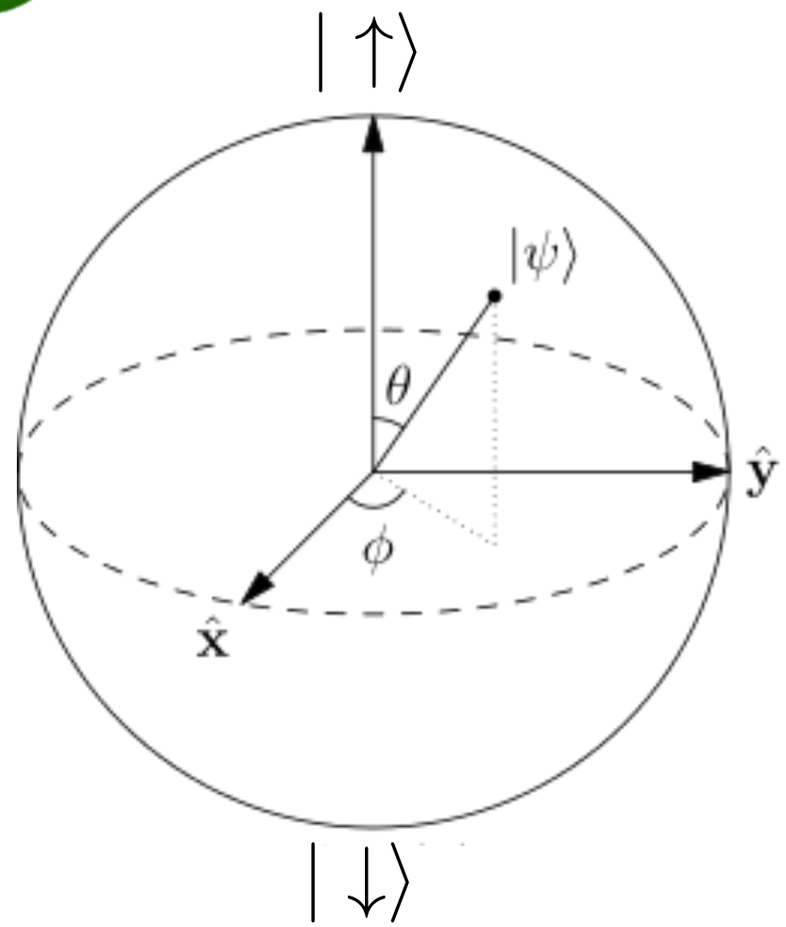
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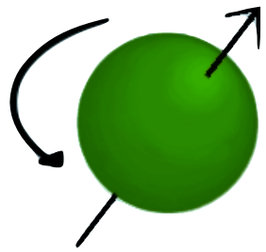
More generally, it is described by a mixed state (density matrix)

$$\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}$$

with $|\vec{r}| \leq 1$ (pure states have $\rho = |\psi\rangle\langle\psi|$), the *Bloch ball*.

Generalized probabilistic theories

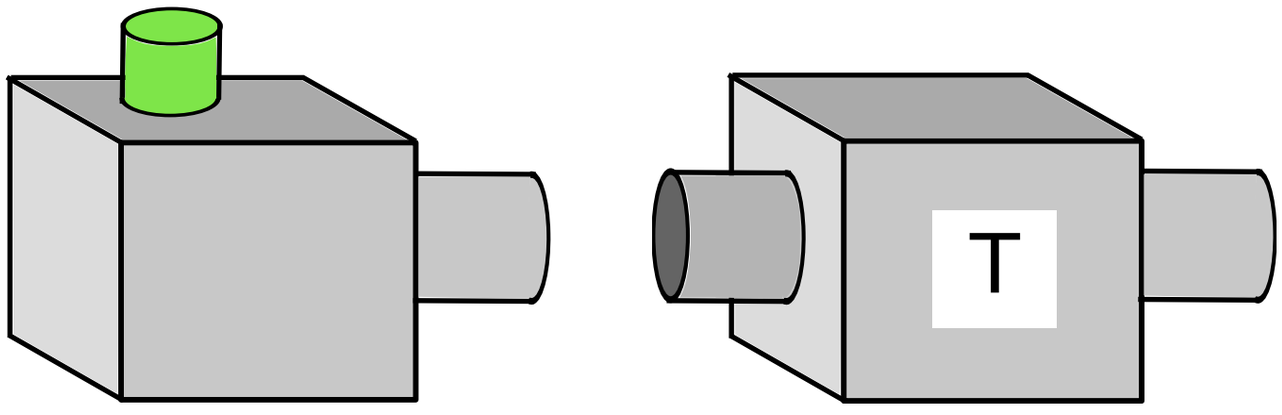
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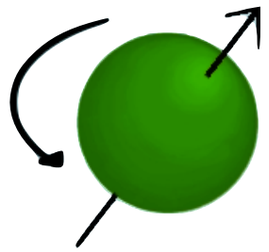


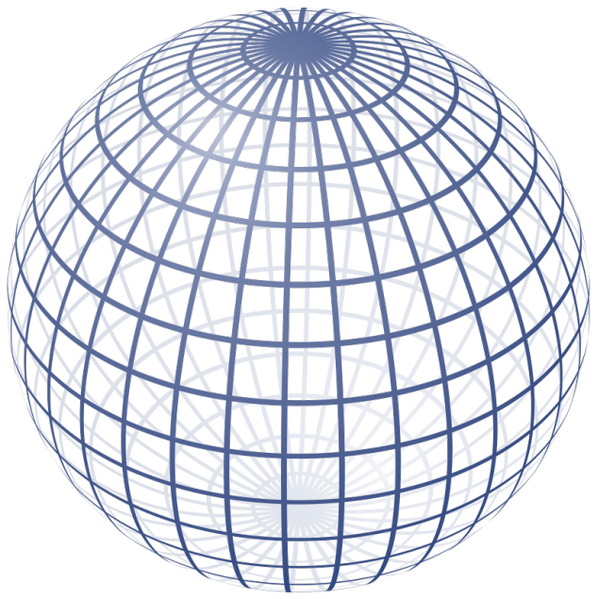


Preparation transformation measurement

Generalized probabilistic theories

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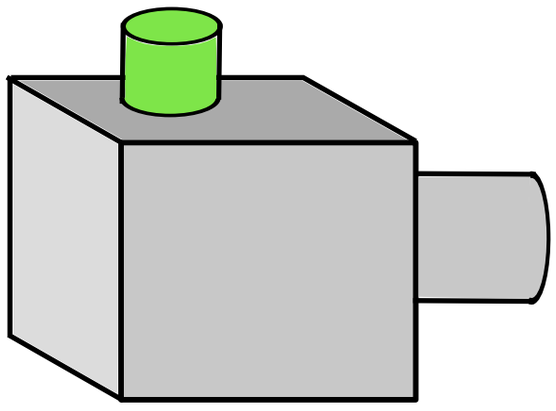


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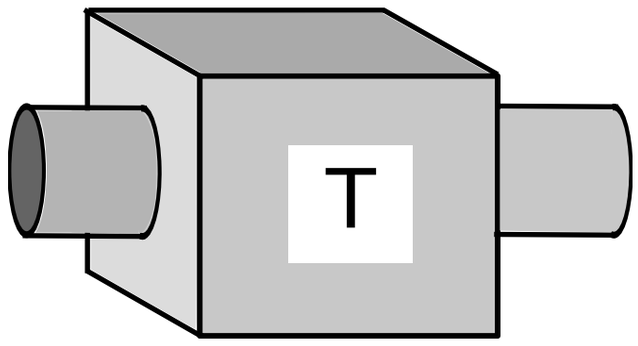
$$\omega \mapsto U\omega U^\dagger.$$

- Measurement in arbitrary spin direction *d*:

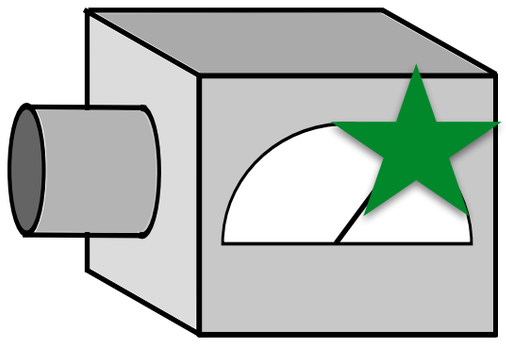
$$\mathrm{Prob}(\uparrow \,|\omega) = \mathrm{Tr}(P_d\,\omega)$$



Preparation



transformation



measurement

Generalized probabilistic theories

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  It is the thing that allows us to determine, *for all possible measurements,* the probabilities of the possible outcomes.

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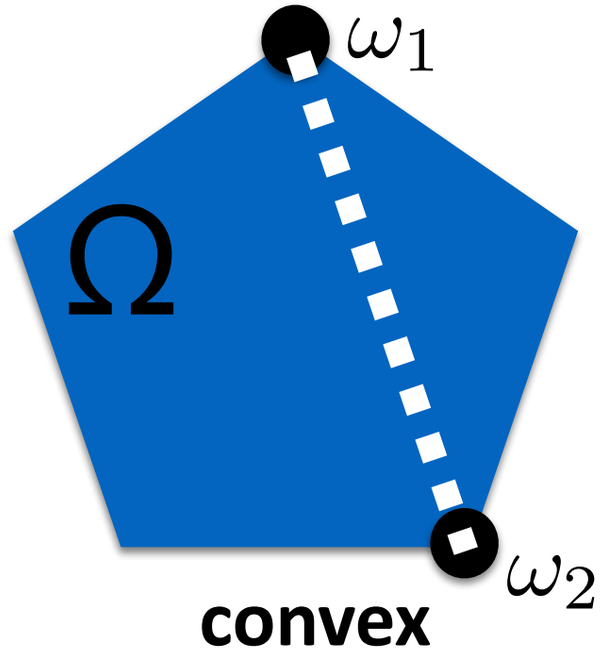
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convex

Generalized probabilistic theories

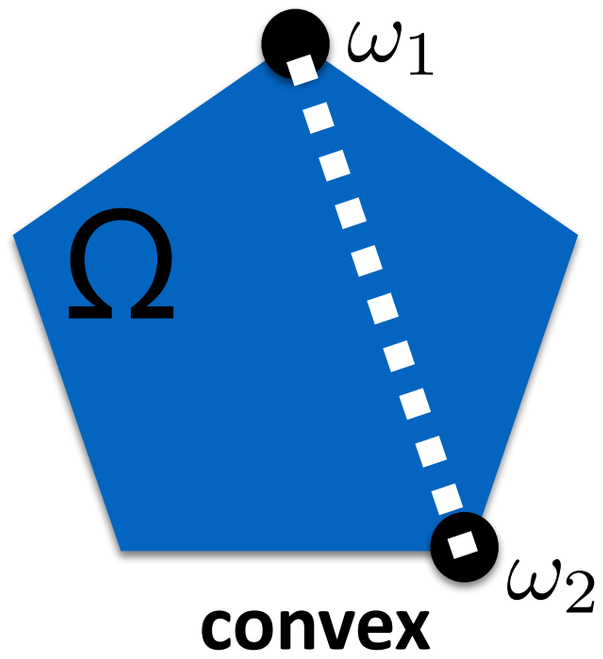
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Generalized probabilistic theories

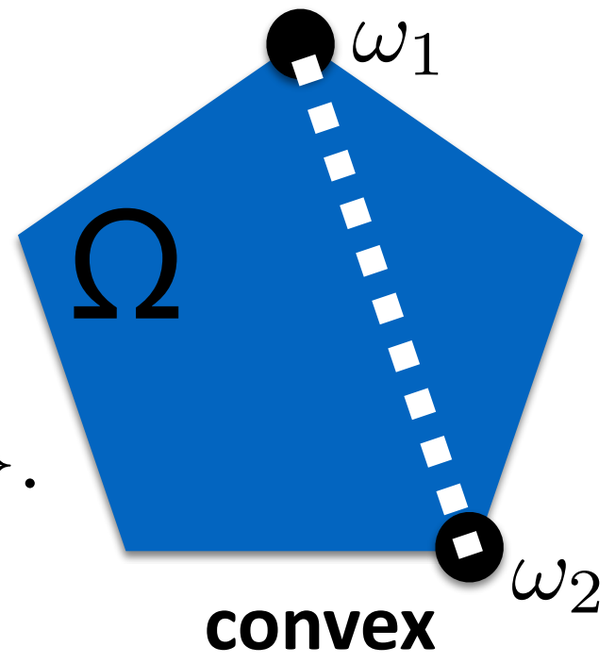
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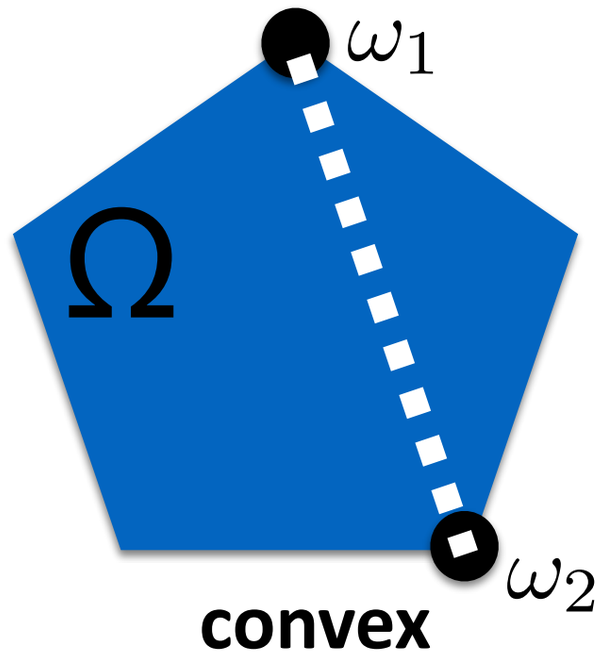
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Generalized probabilistic theories

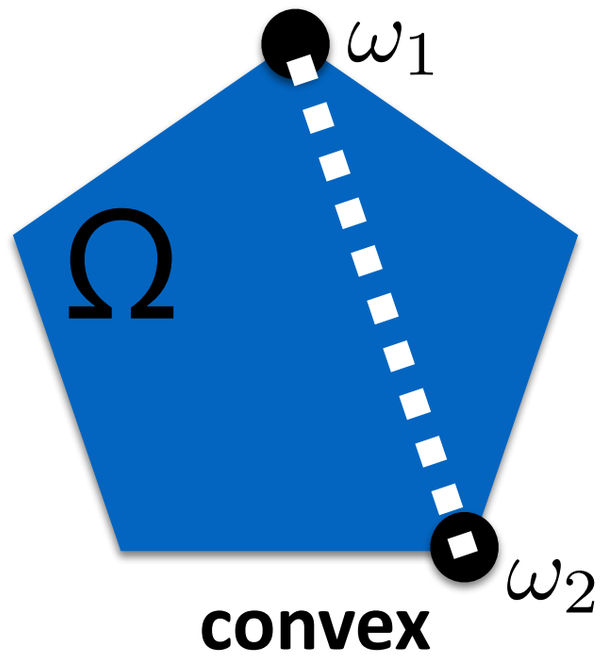
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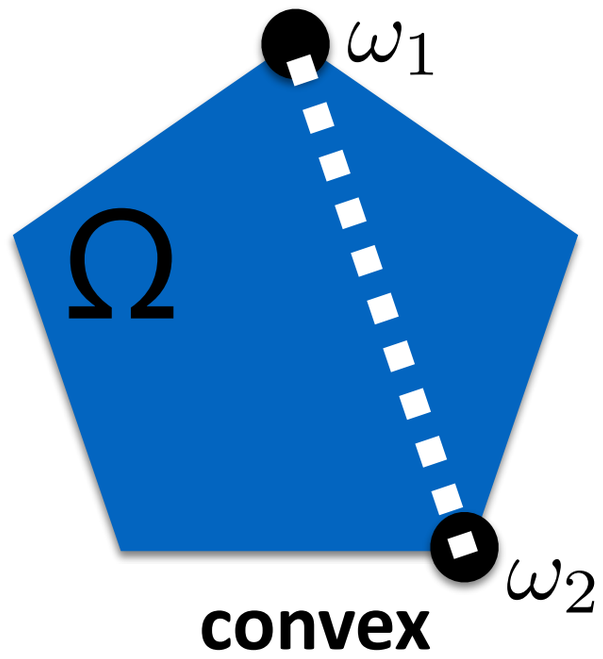


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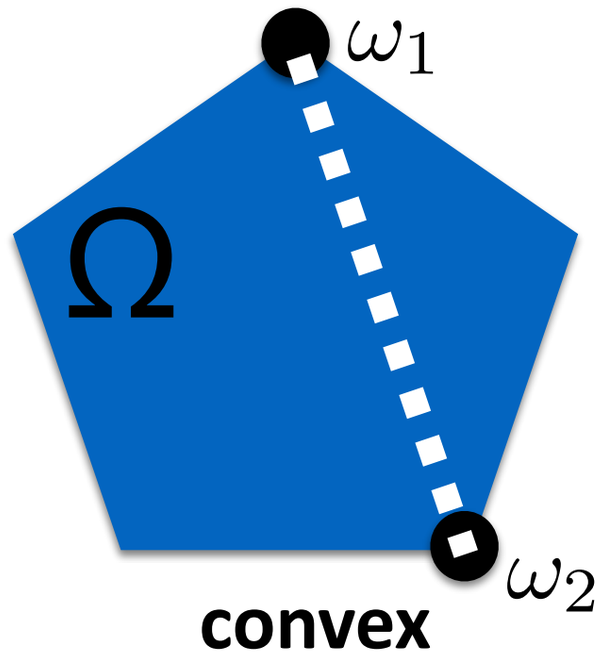
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QT: POVMs (positive operator-valued measures),
$e_i(\omega) = \mathrm{tr}(E_i\omega)$.



Generalized probabilistic theories



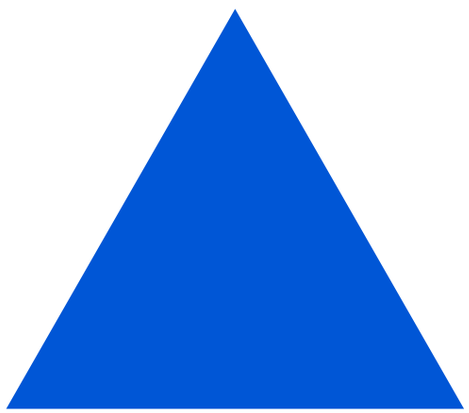
classical bit



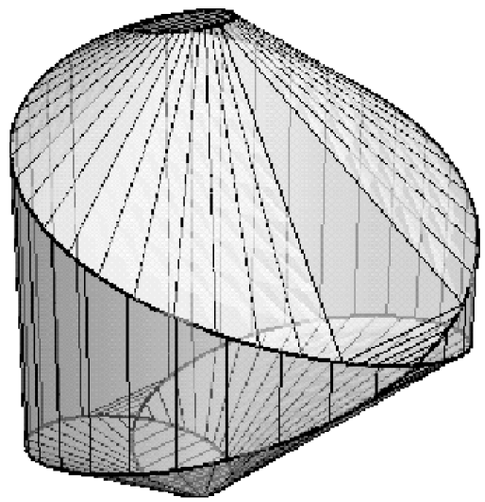
quantum bit



"gbit"



Classical trit
(3-level-system)



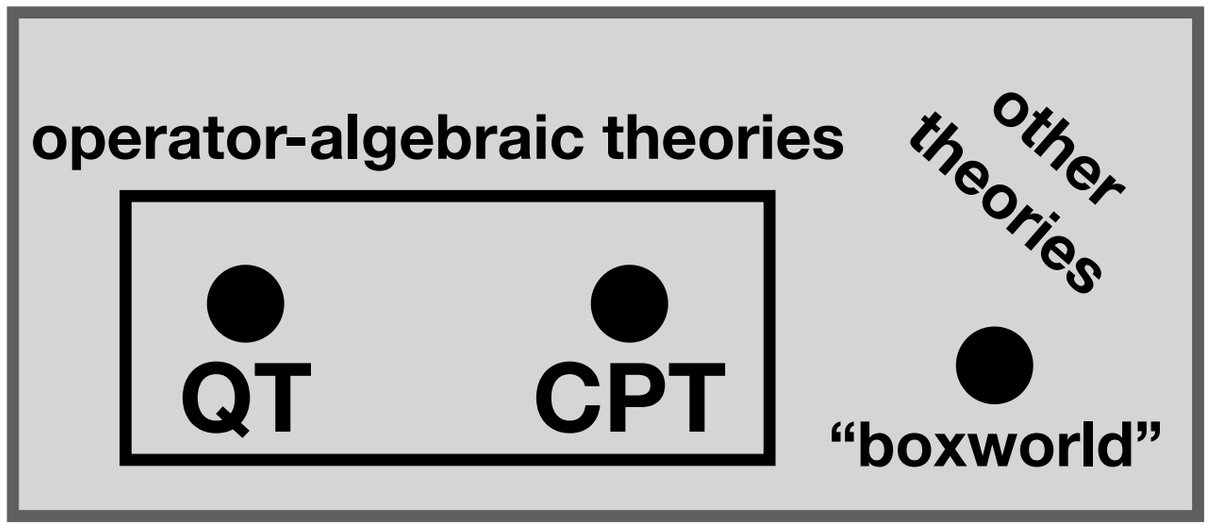
Quantum trit:
8D and complicated!



Arbitrary convex state space

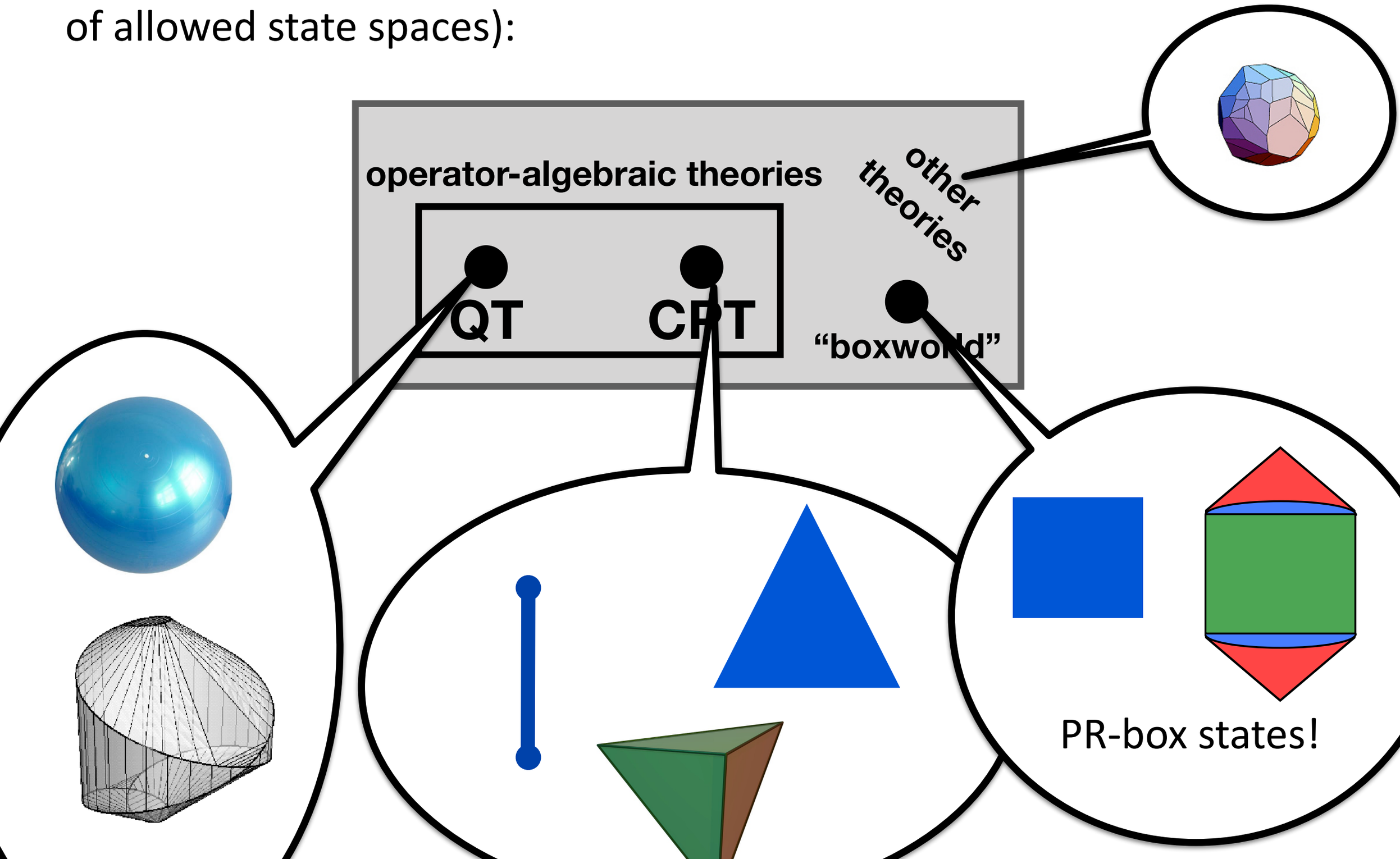
Generalized probabilistic theories

There is a large landscape of state spaces, or *theories* (collections of allowed state spaces):



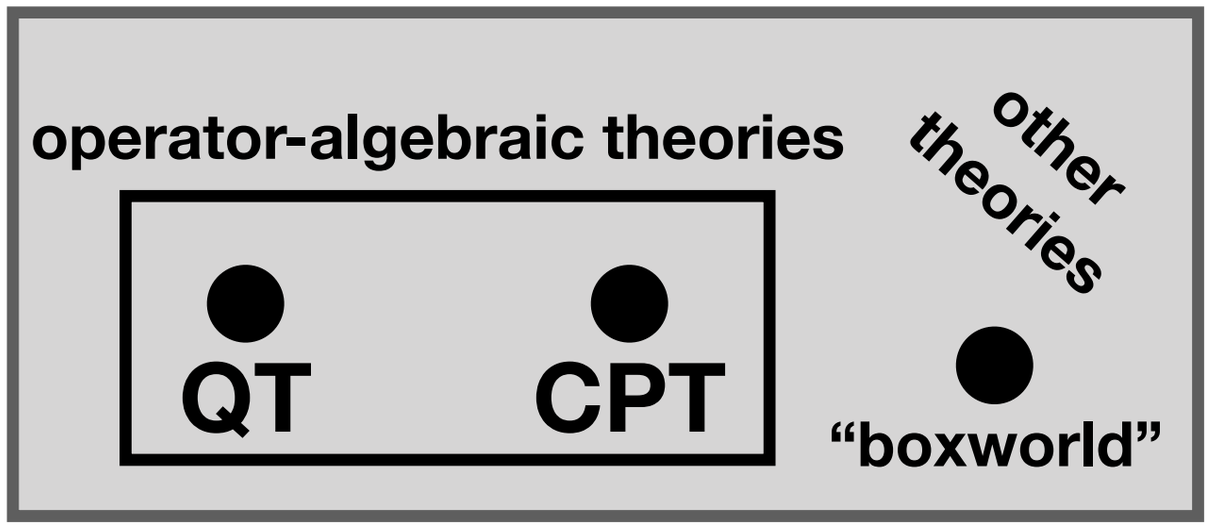
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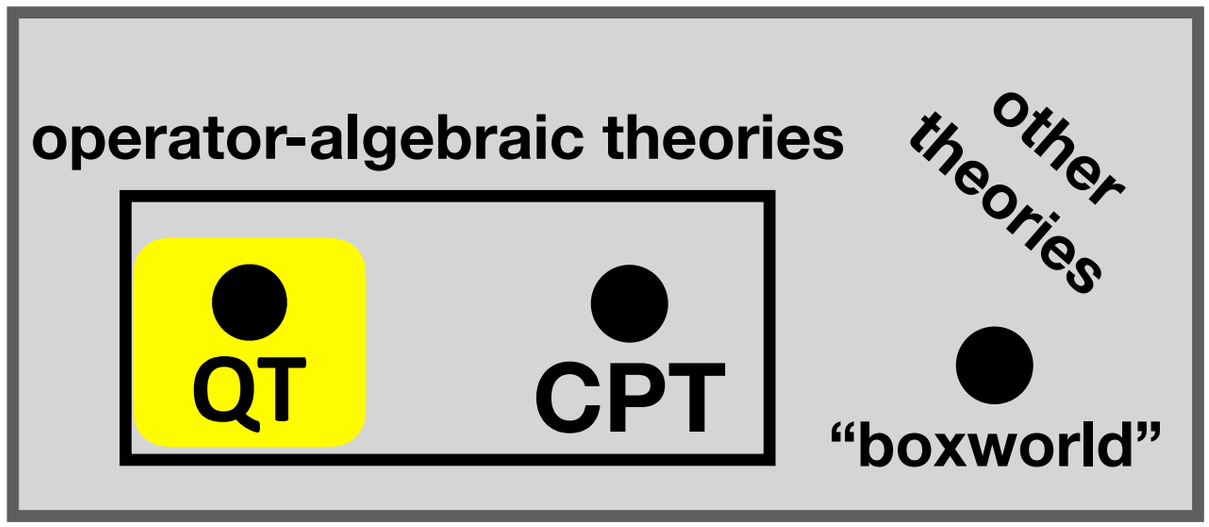
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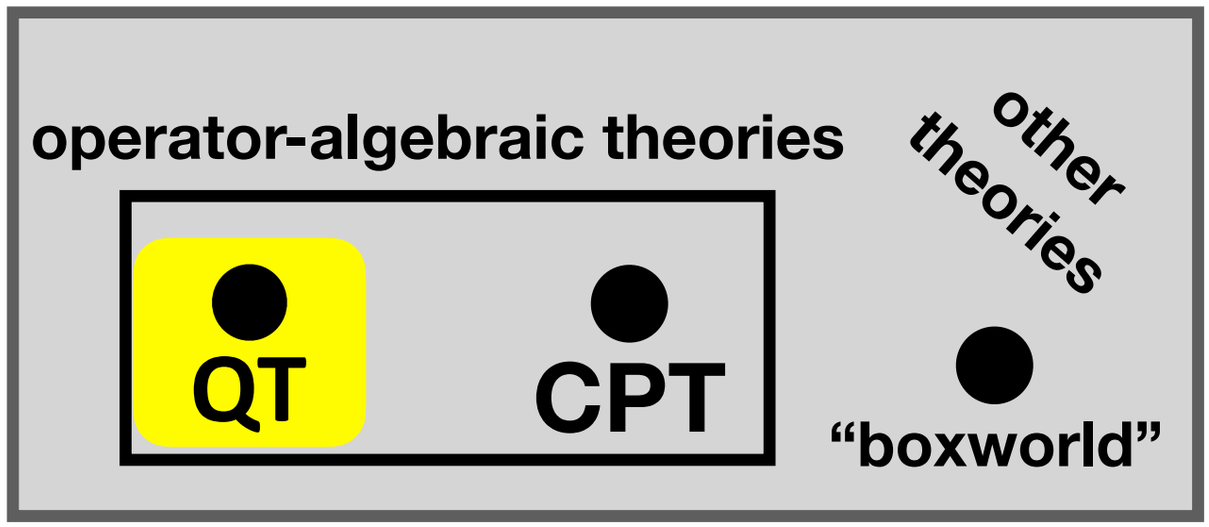
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Goal: Find a small set of simple physical / information-theoretic **principles** that singles out QT uniquely.

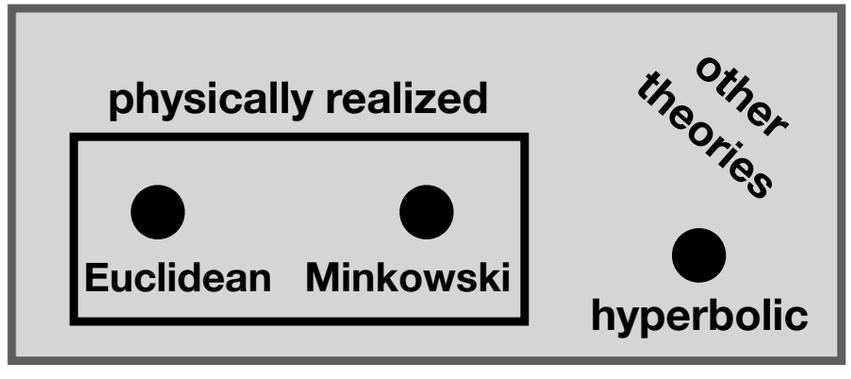
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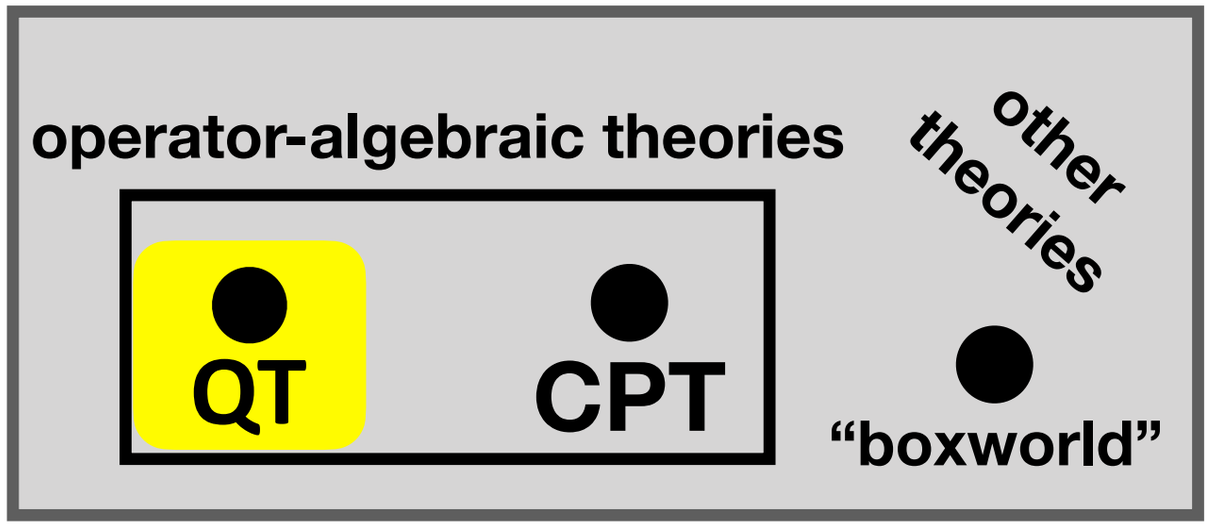


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Role model: Einstein's **Relativity Principle** and **Light Postulate** determine Minkowski spacetime.

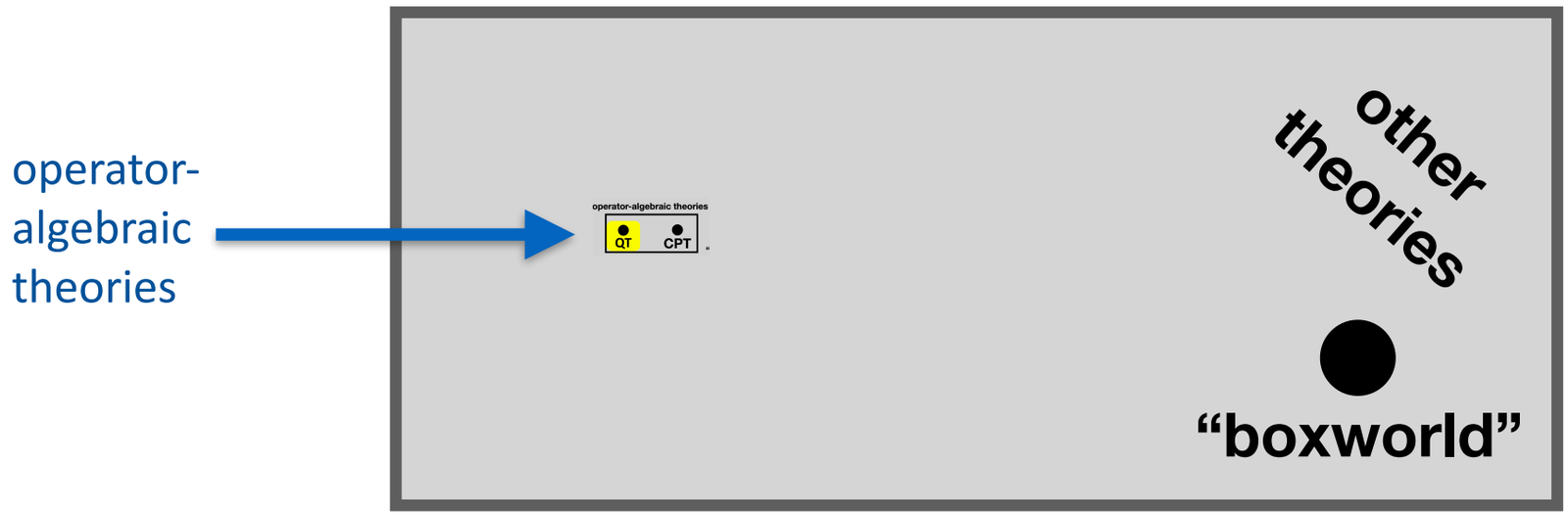


Some comments on work from the philosophy community



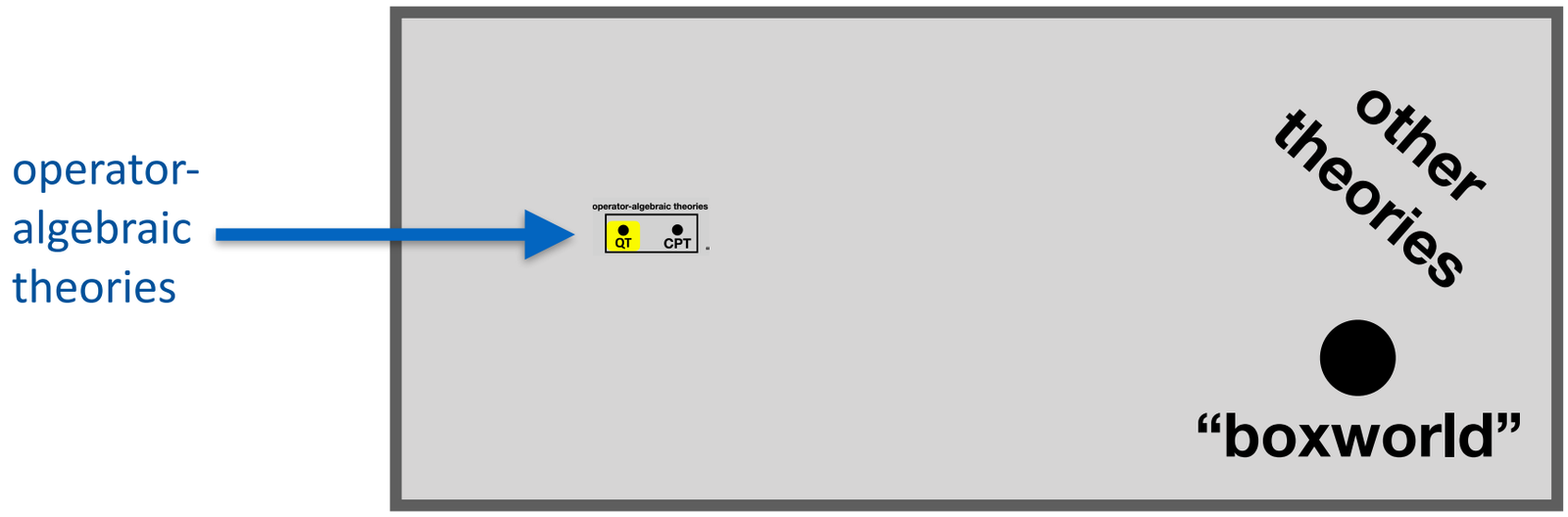
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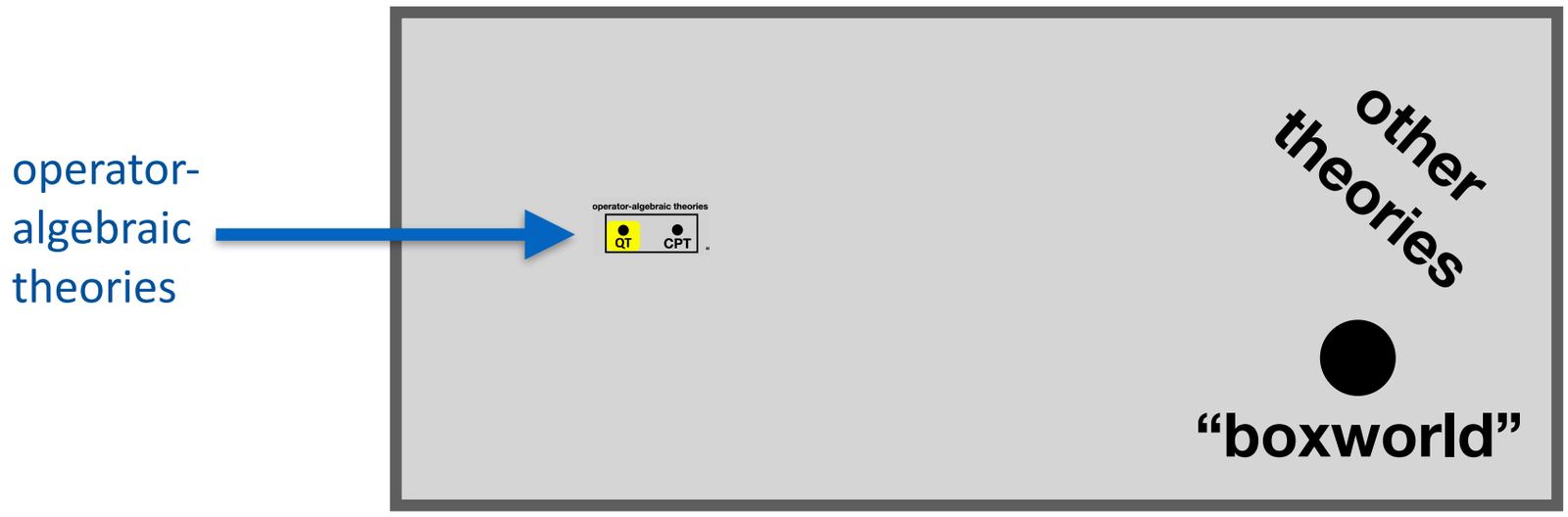
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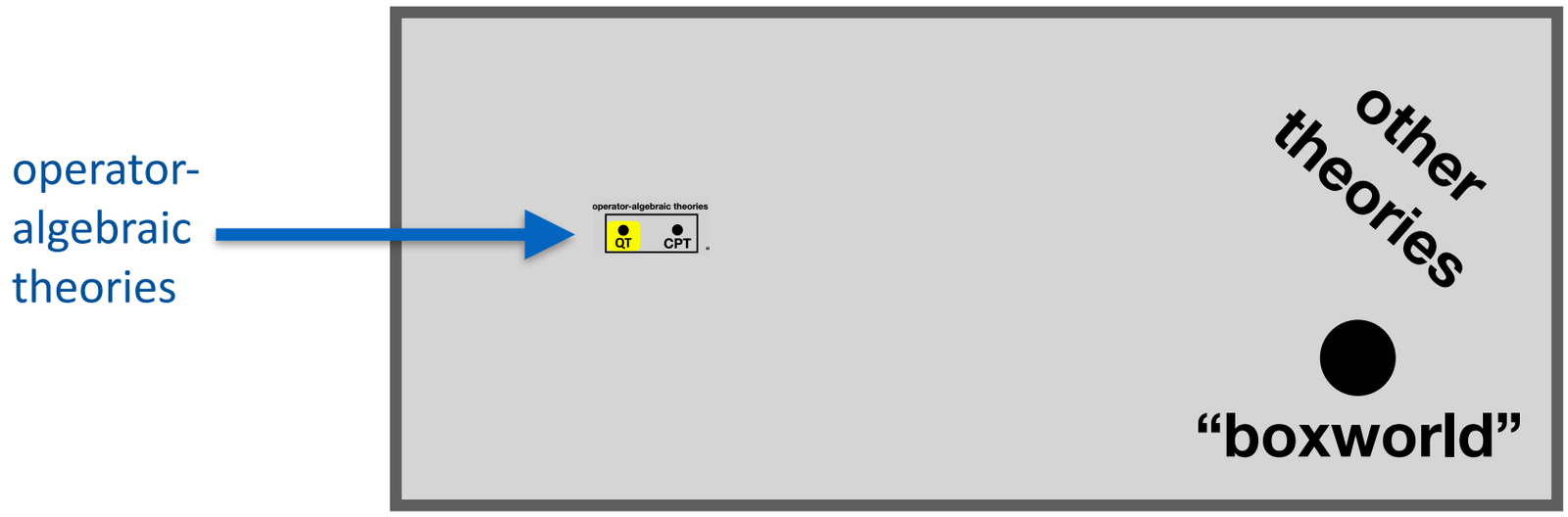
Hence, most physicists were unimpressed by the "CBH"-reconstruction

R. Clifton, J. Bub, H. Halvorson, *Characterizing quantum theory in terms of information-theoretic constraints,* Found. Phys. **33**, 1561-1591 (2003)

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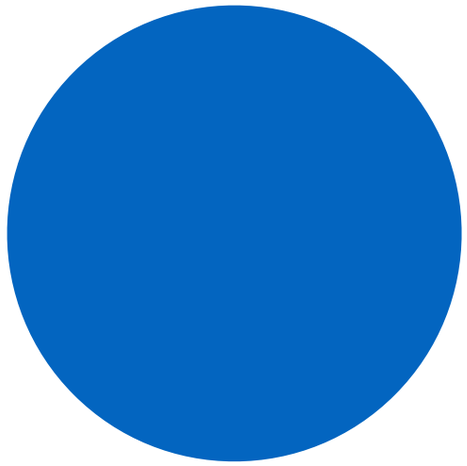
H. Brown and C. Timpson, *Why special relativity should not be a template for a fundamental reformulation of quantum mechanics*, Physical Theory and its Interpretation, W. Demopoulos and I. Pitowsky (eds.), Springer, 2006.

Main objection: reconstructions leave explanatory work unfinished.

But many physicists happy with structural rather than mechanistic explanations.

Generalized probabilistic theories

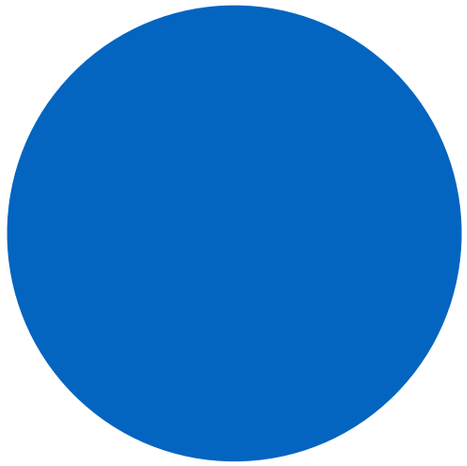
Physical properties depend (strongly!) on the **shape** of the state space.





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quantum
bit (over $\mathbb{R}$)



"gbit"
(neither classical nor quantum)

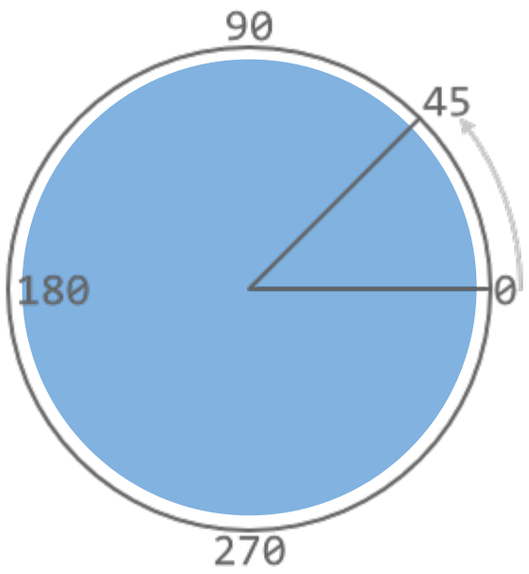
$$\rho = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1-r_3 \end{pmatrix},$$

$$r_2 := 0.$$

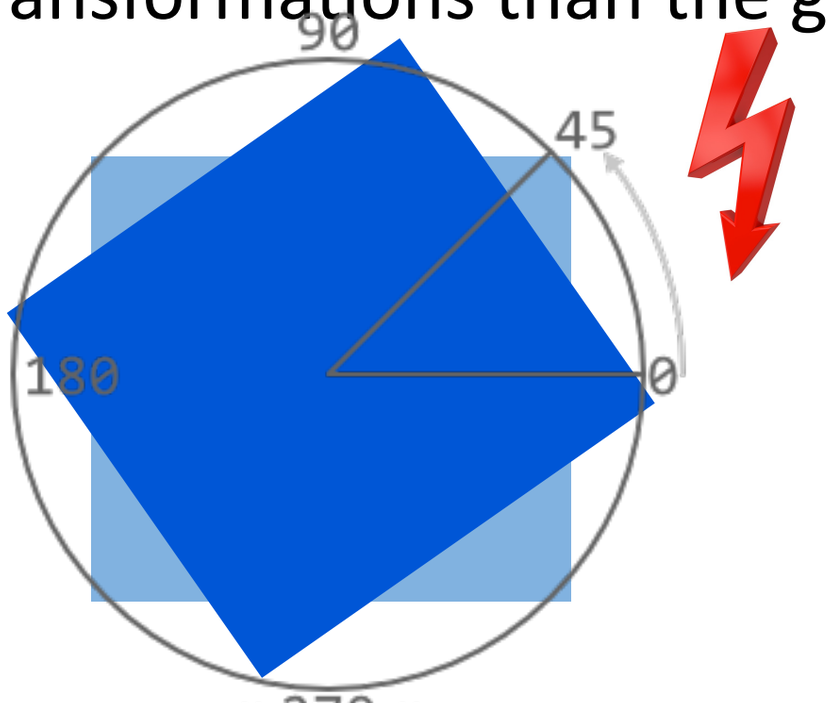
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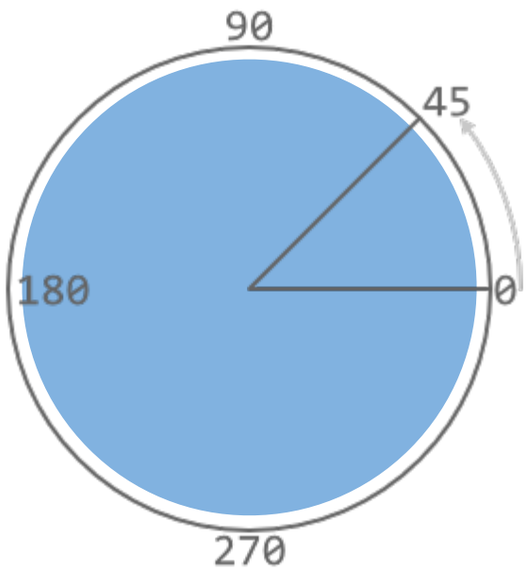


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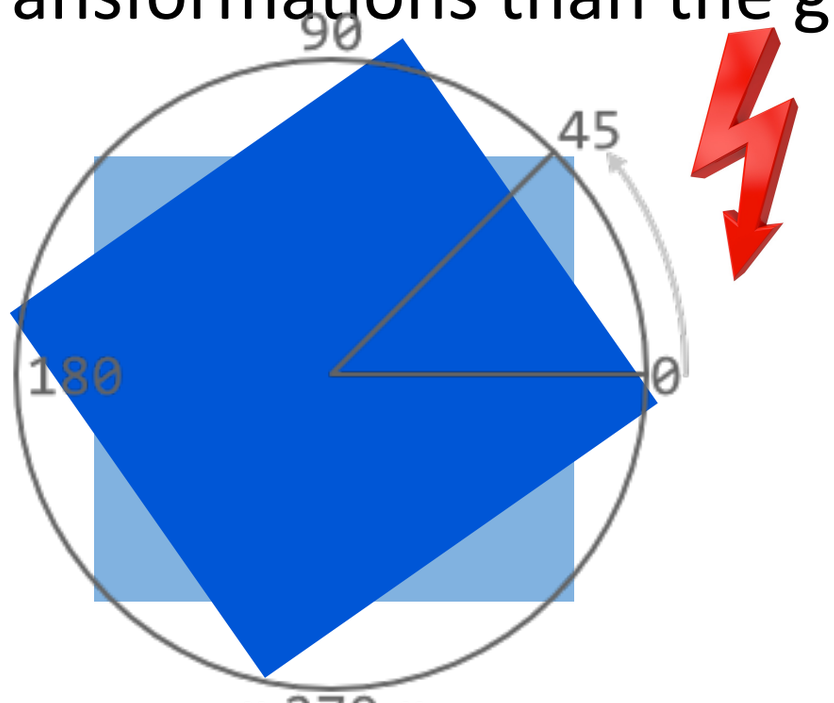
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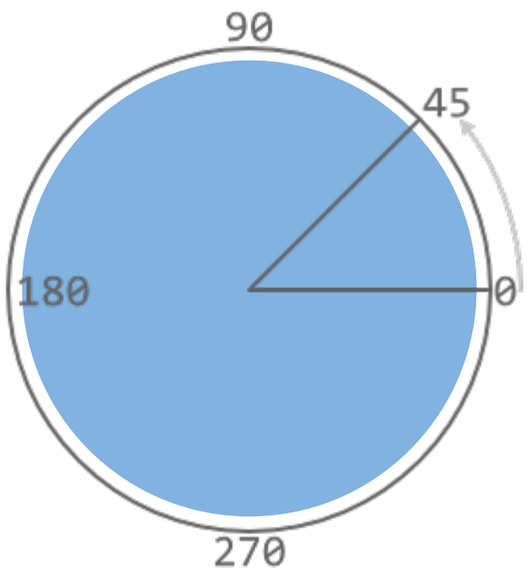
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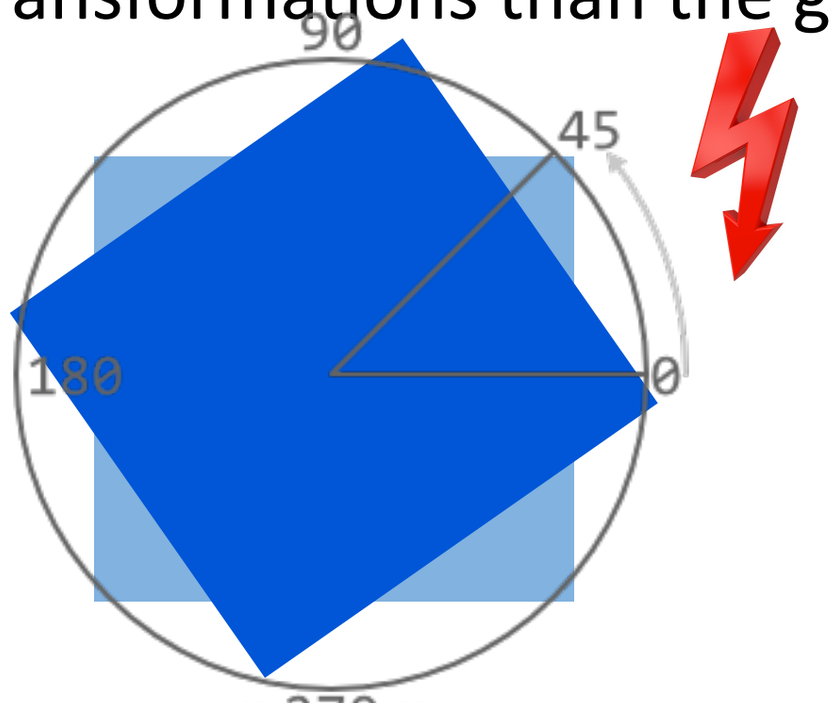
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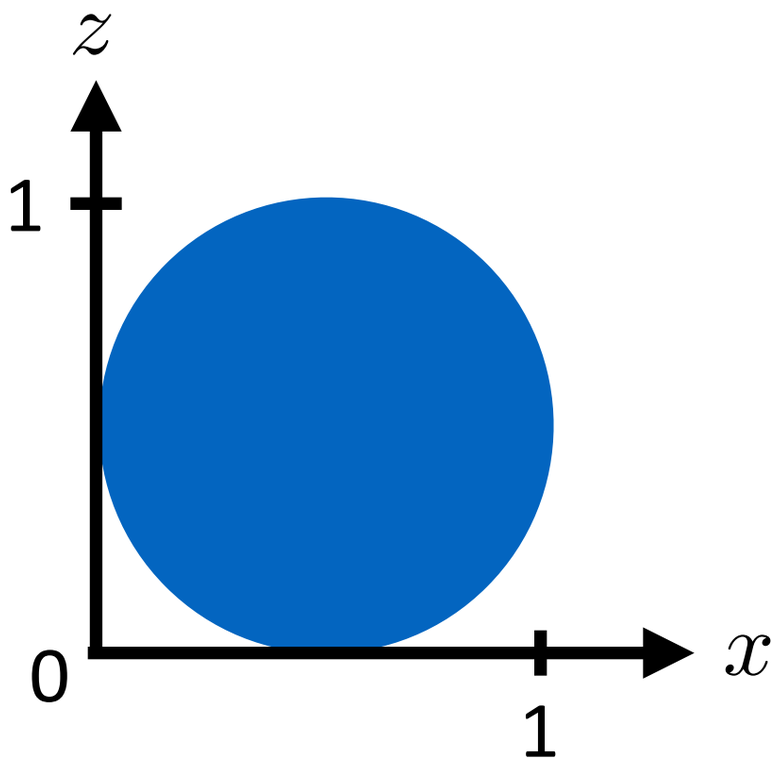
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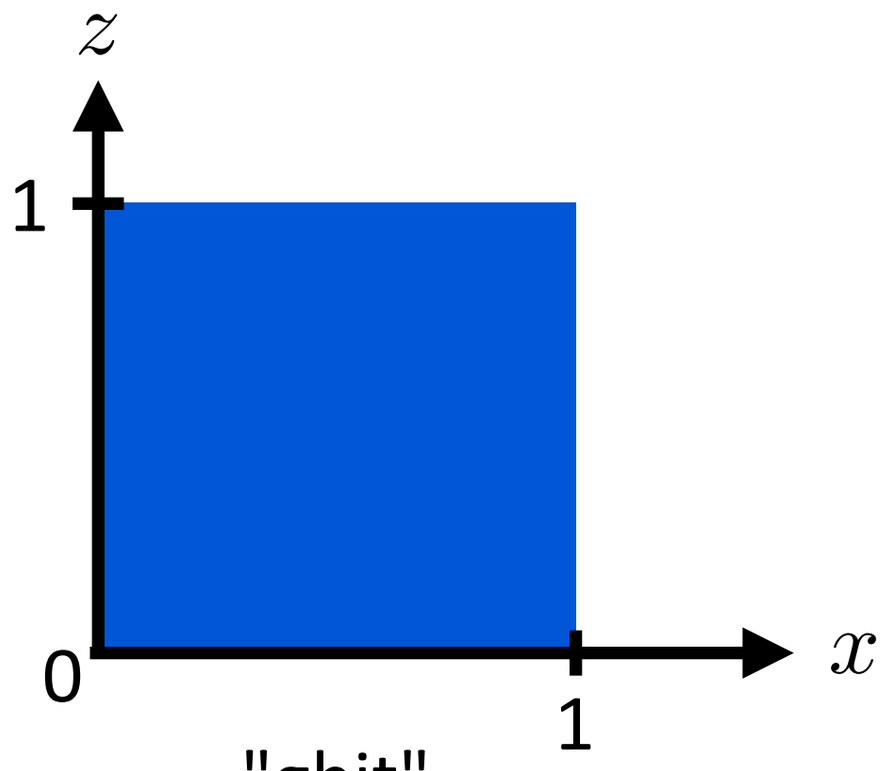
In particular, the qubit admits **continuous reversible time evolution**, but the gbit does not.

Generalized probabilistic theories

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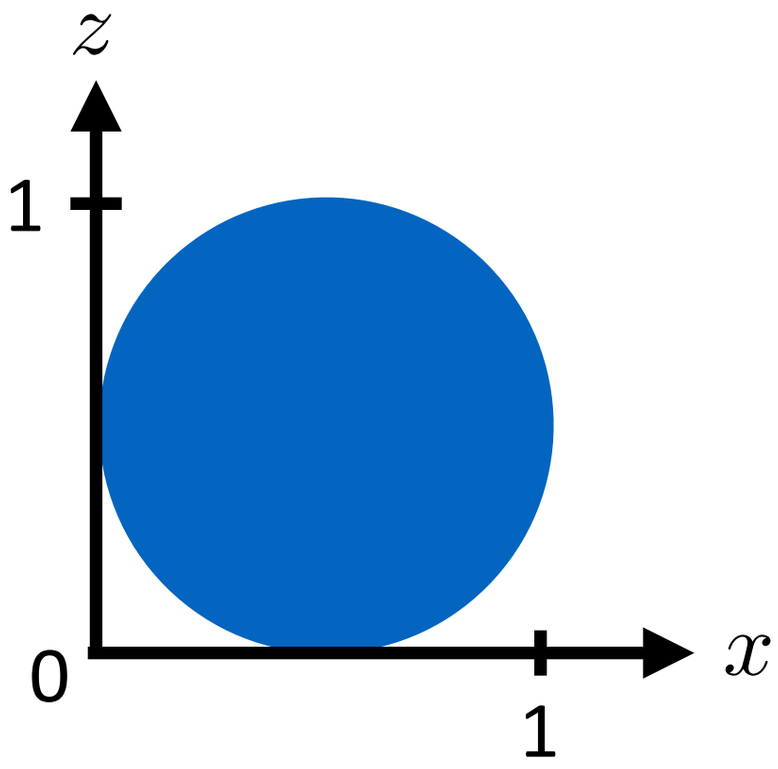
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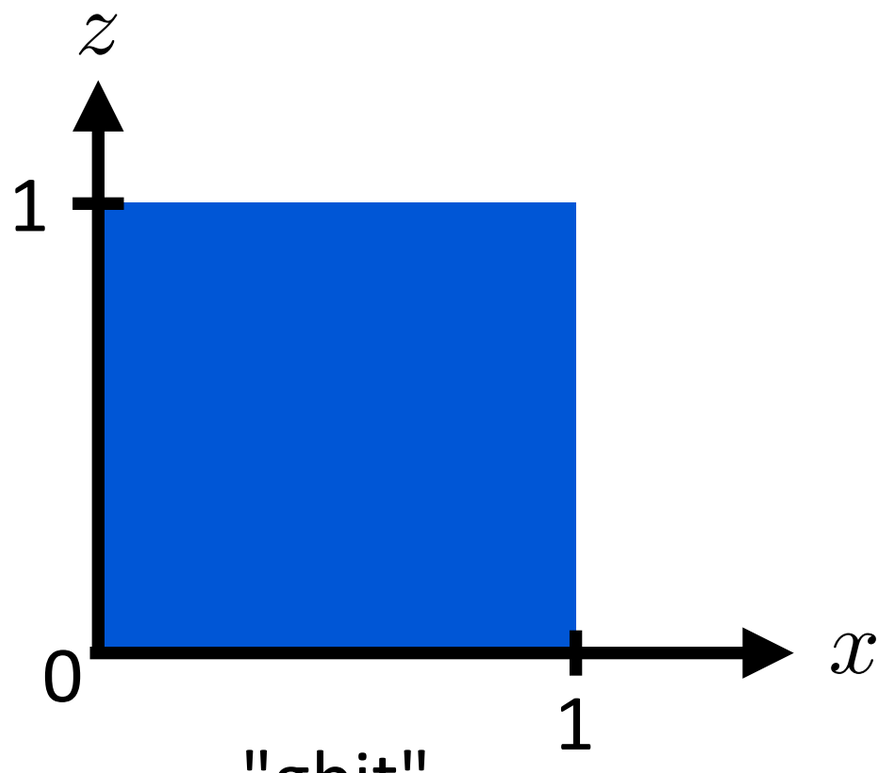
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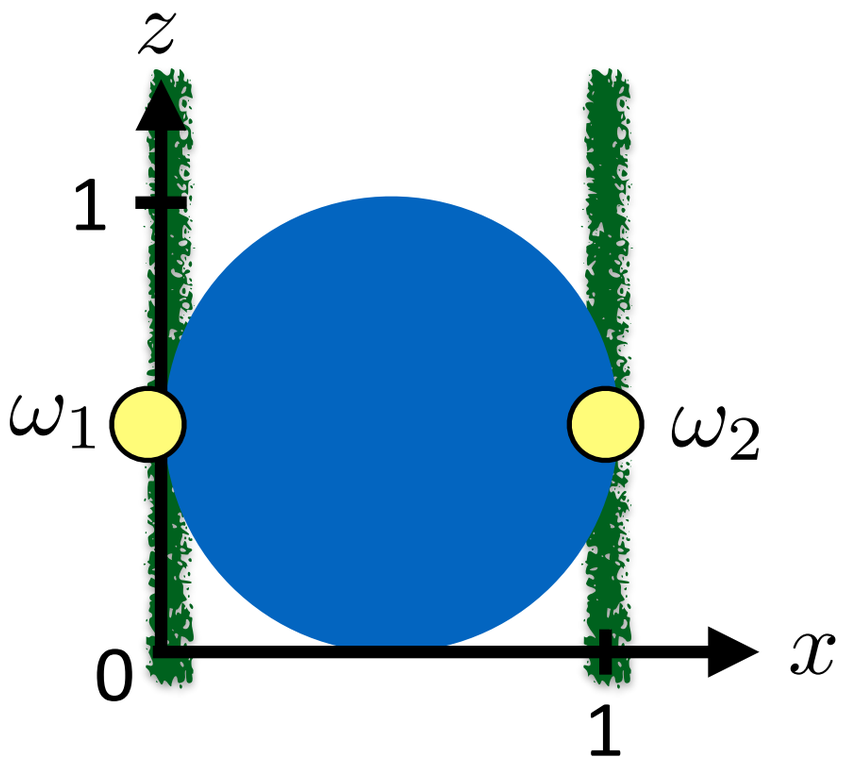
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In both cases, can think of a 2-outcome “X”-measurement with

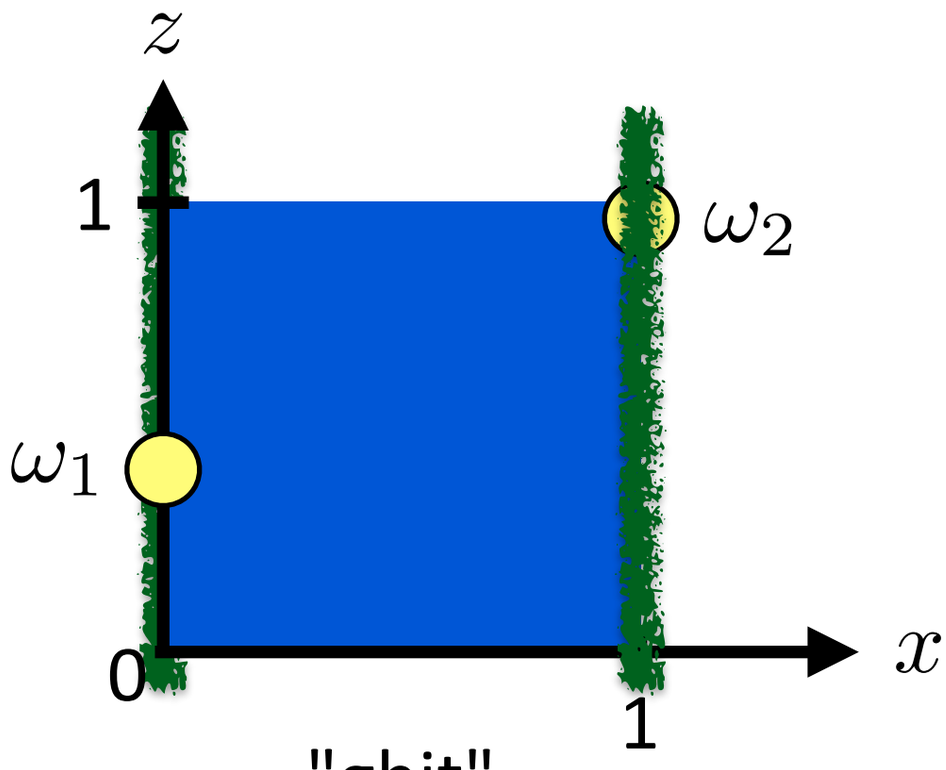
$$\mathrm{Prob}(\mathrm{yes}|X,\omega) = x, \quad \mathrm{Prob}(\mathrm{no}|X,\omega) = 1 - x.$$

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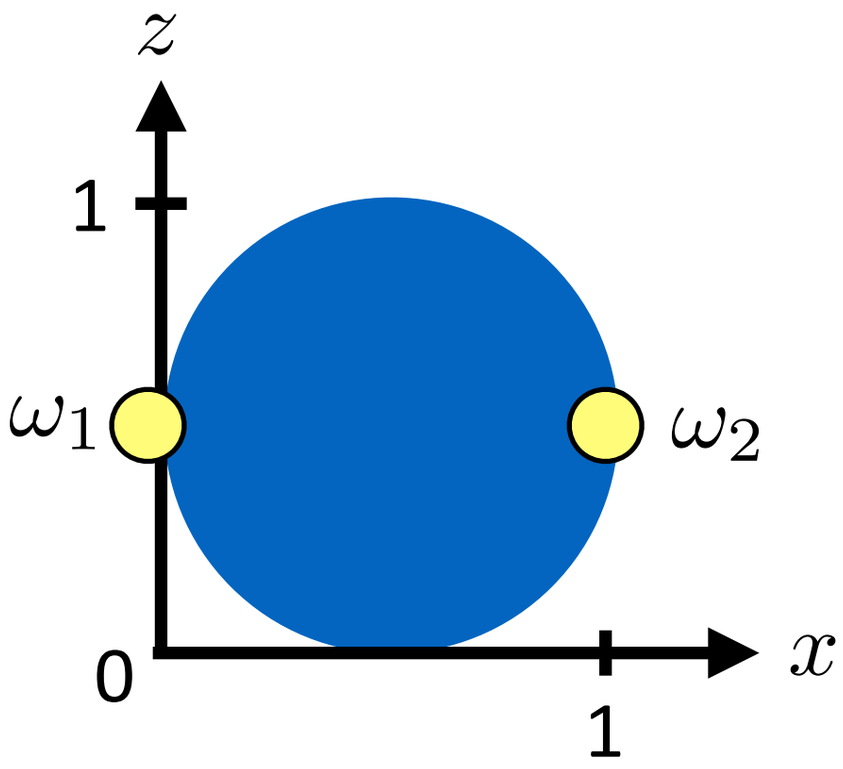
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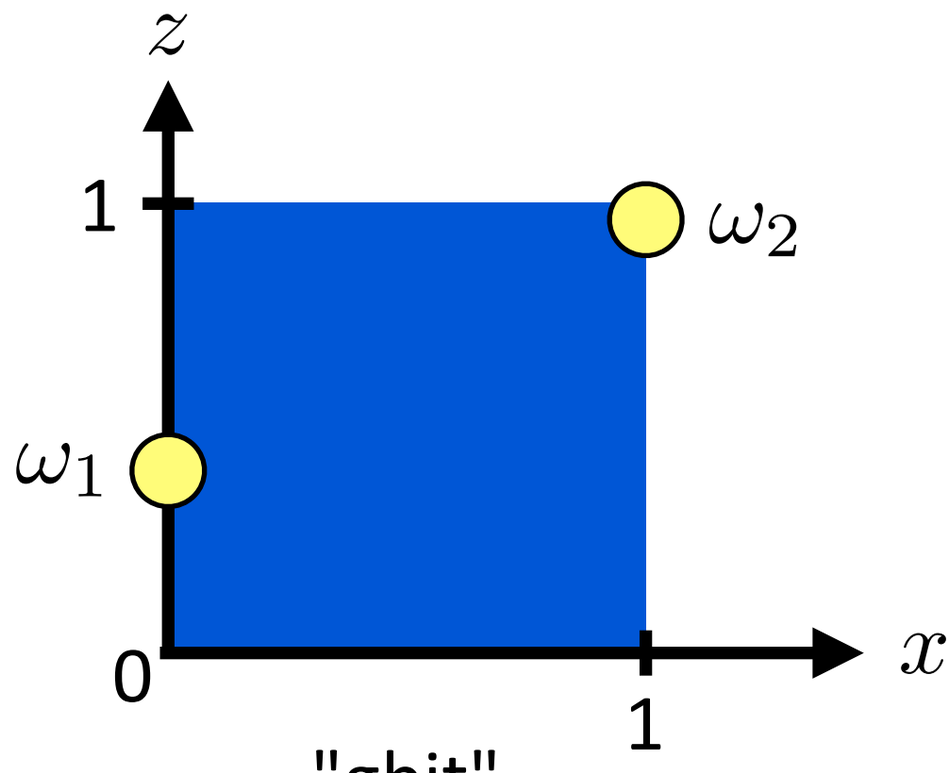
Distinguishes two states ω_1, ω_2 perfectly (deterministically).

Generalized probabilistic theories

Physical properties depend (strongly!) on the **shape** of the state space.



quantum bit



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In both cases, the maximal number of perfectly distinguishable states is 2:

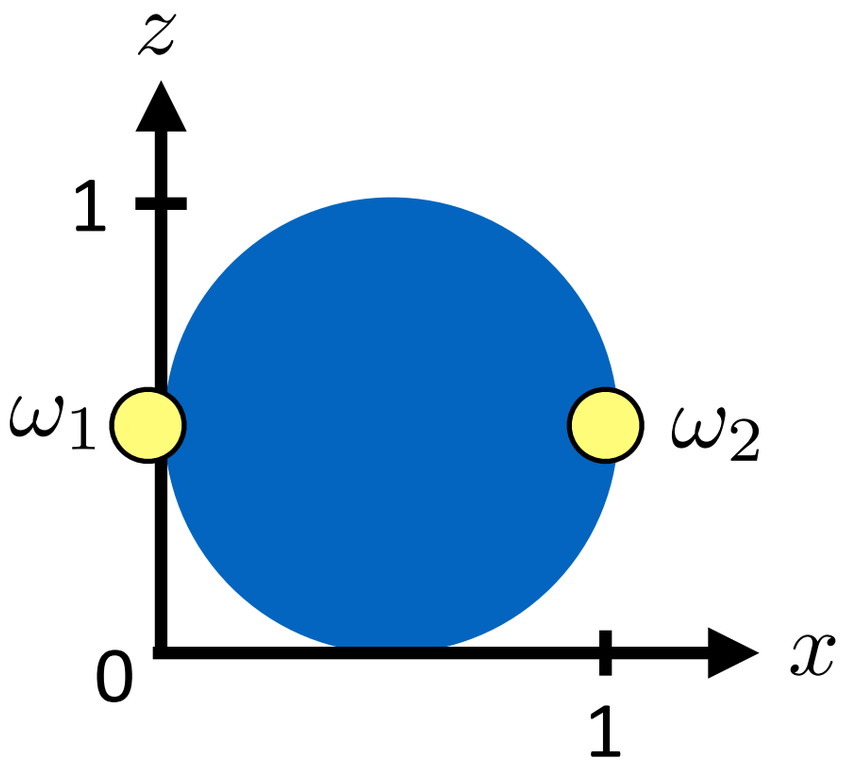
$\omega_1, \ldots, \omega_n$ states, $e_1, \ldots, e_n$ effects summing to unit effect,

$e_i(\omega_j) = \delta_{ij} \quad \Rightarrow n \leq 2.$

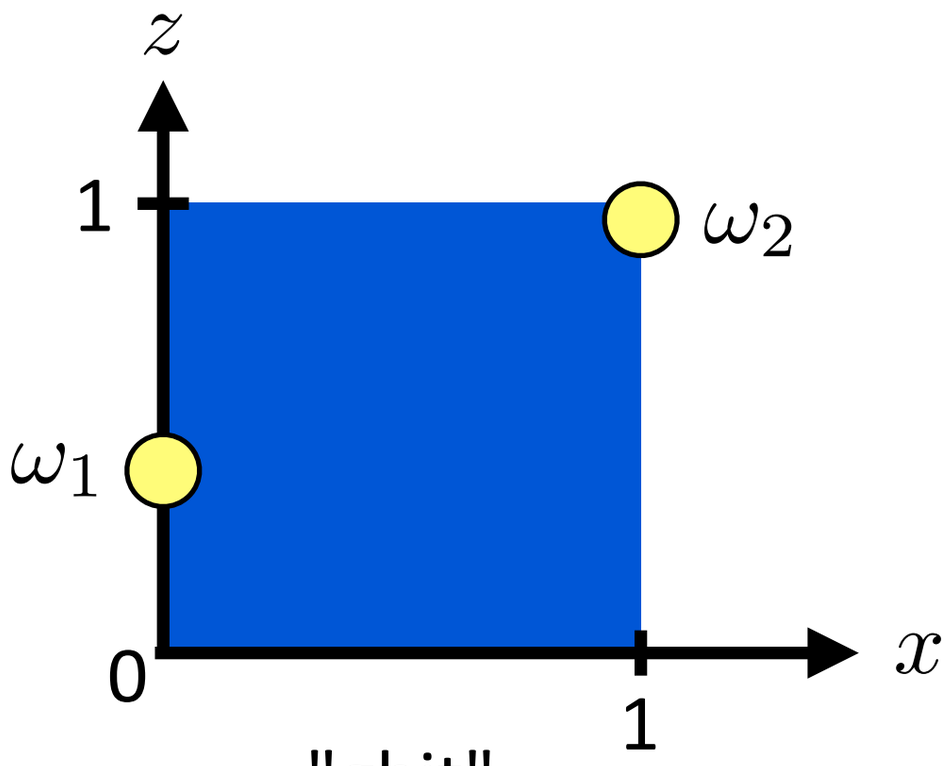
Thus, these are both "**bits**".

Generalized probabilistic theories

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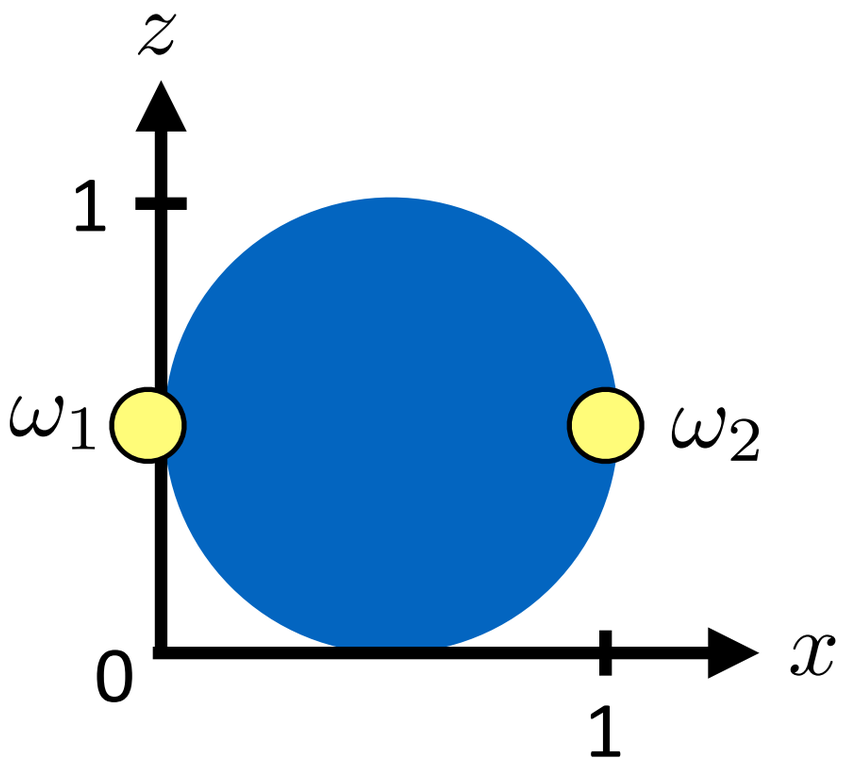
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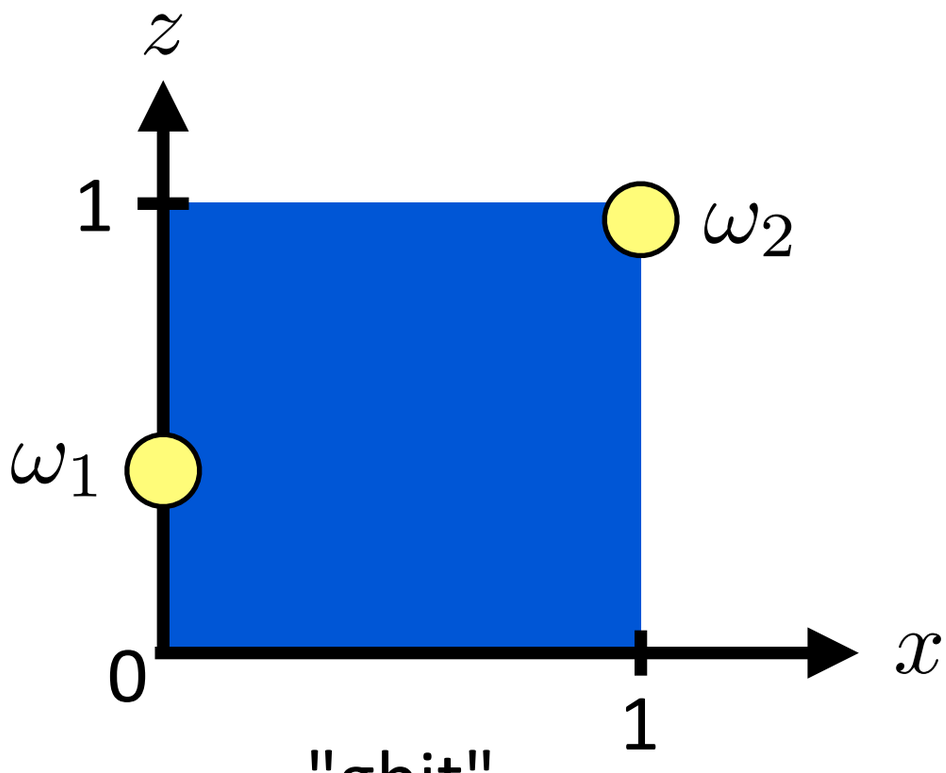
Both feature "**complementarity**": cannot predict X and Z simultaneously.

Generalized probabilistic theories

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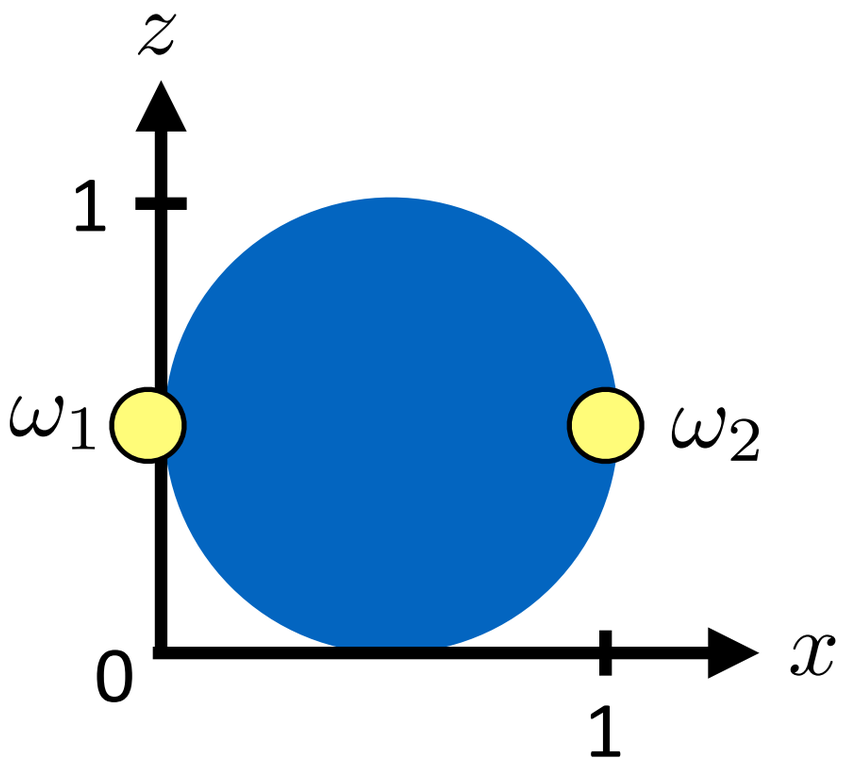
However, the qubit features **uncertainty relations**, but not the gbit.

Qubit: $\mathrm{Prob}(\mathrm{yes}|X,\omega) = 0 \text{ or } 1 \;\Rightarrow \mathrm{Prob}(\mathrm{yes}|Z,\omega) = \frac{1}{2}.$

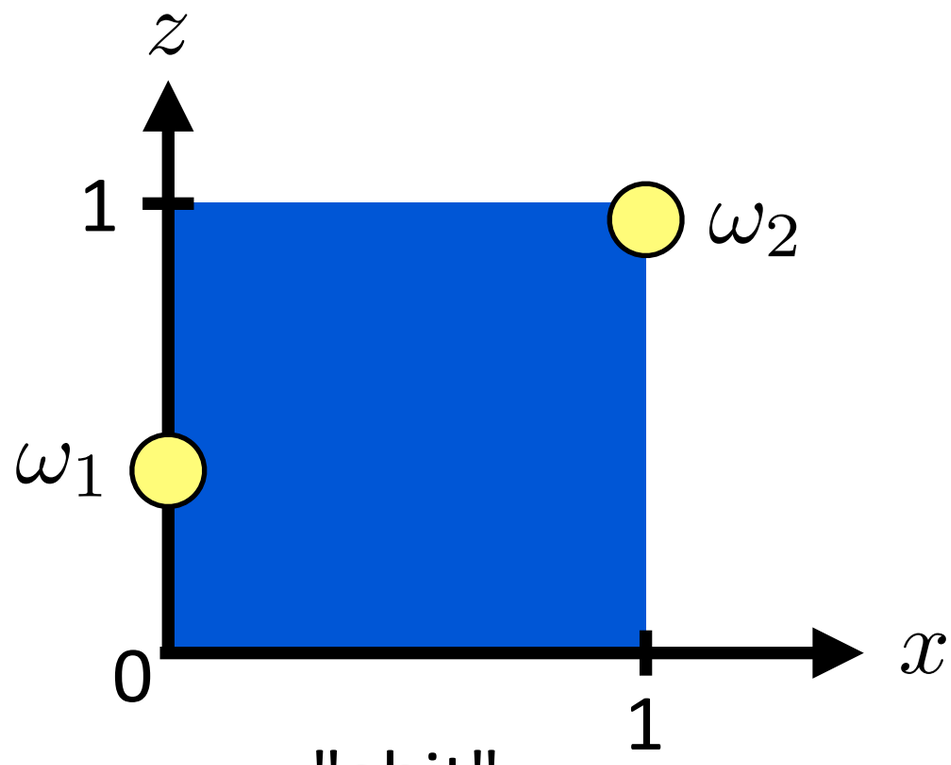
Gbit: results of both measurements can be simultaneously predetermined.

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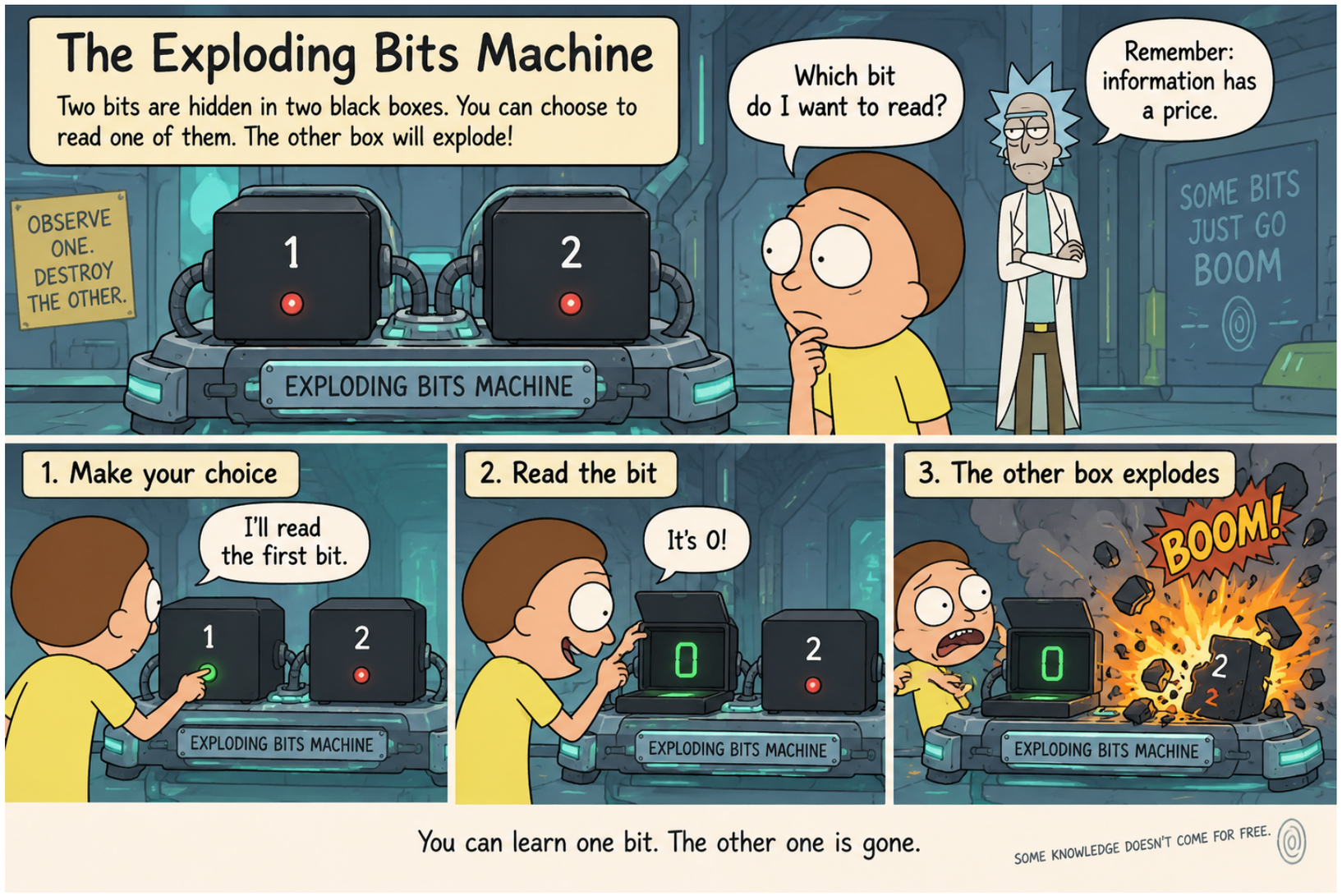
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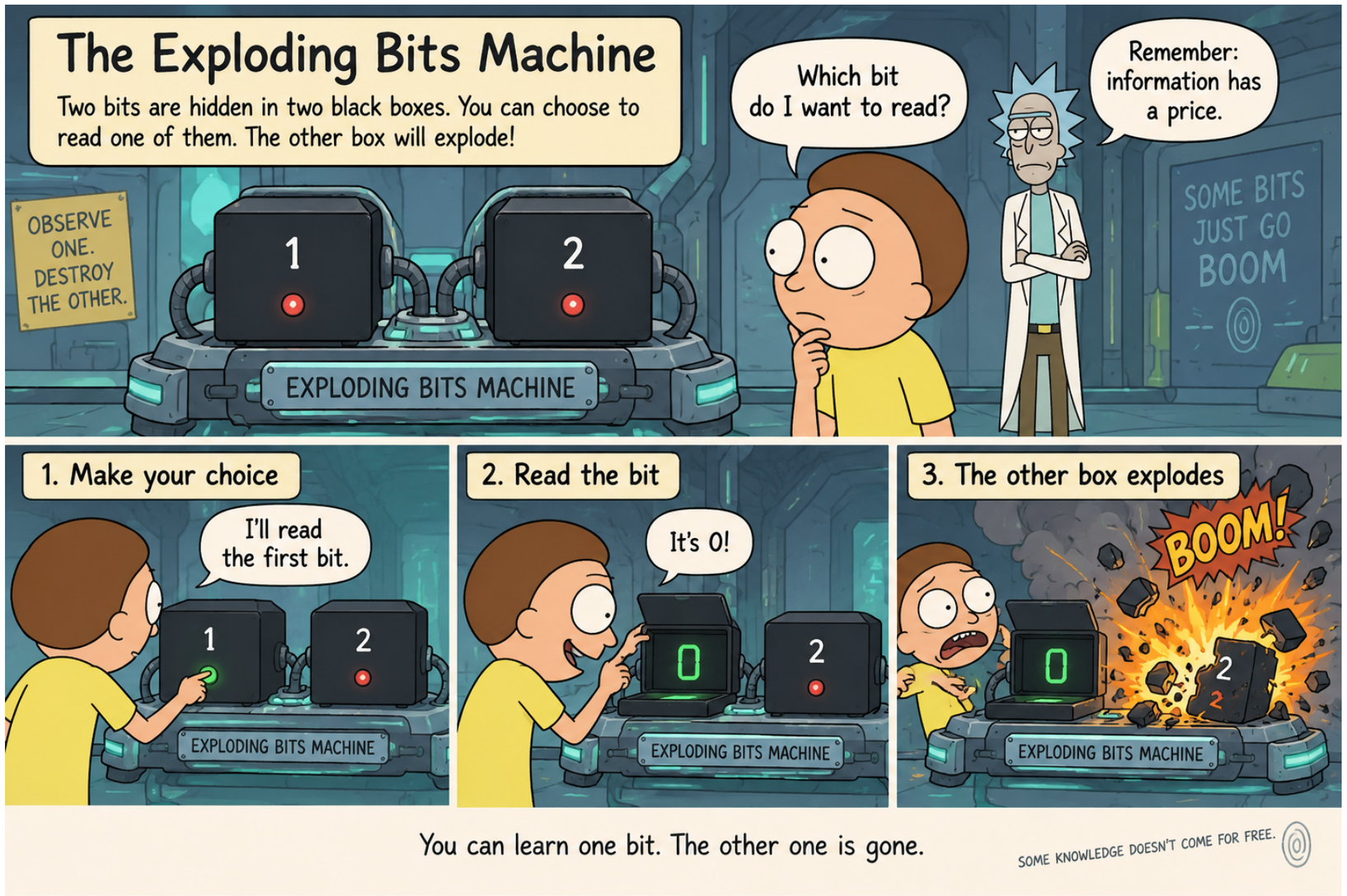
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In some sense, uncertainty relations are “a prize to pay” for Nature to admit of continuous reversible time evolution on *discrete* systems (such as bits)!

Classical simulation of the gbit (sort of...)

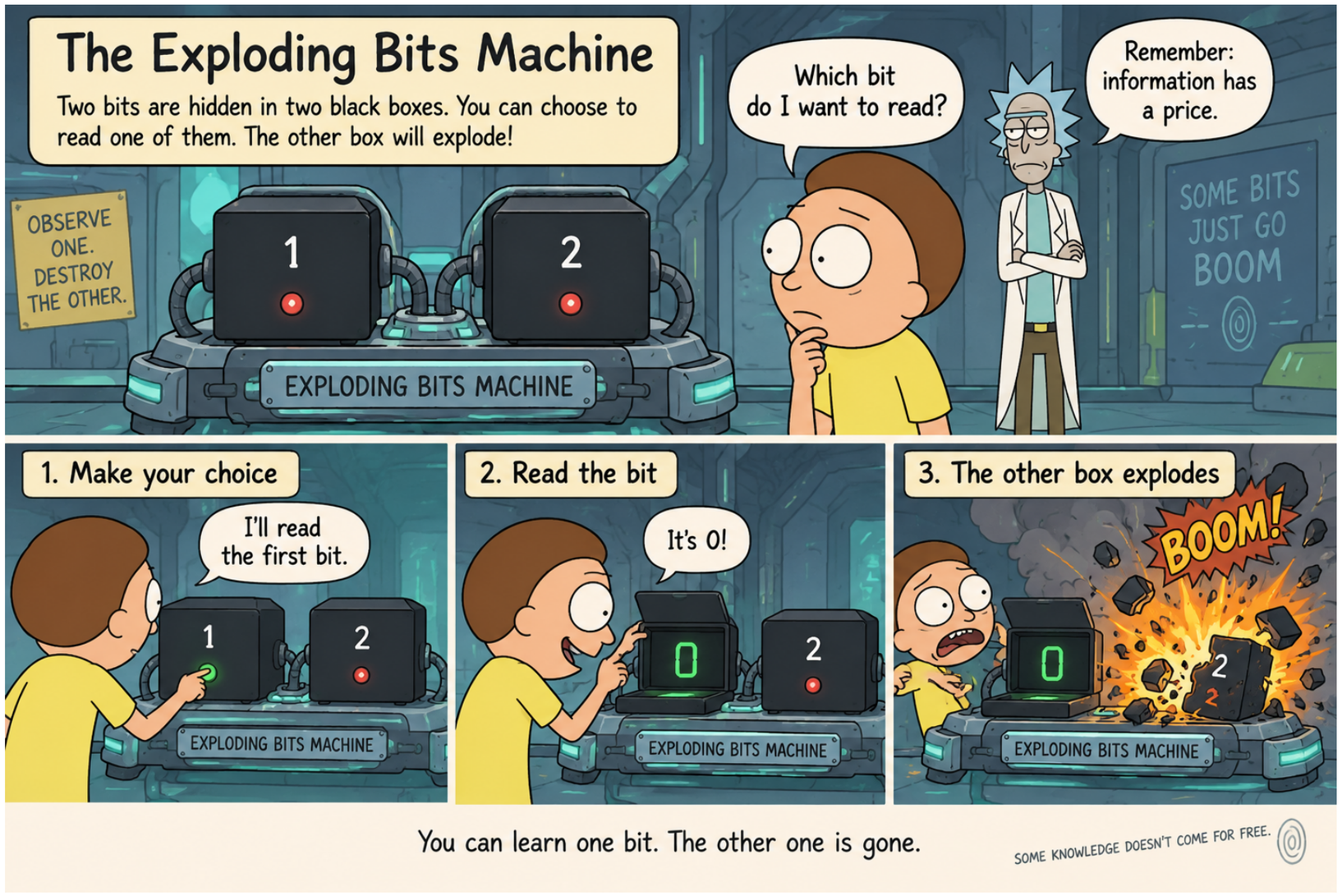


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Denote by $P_{i,j}$ that bit 1 has been prepared to be *i*, and bit 2 to be *j*.

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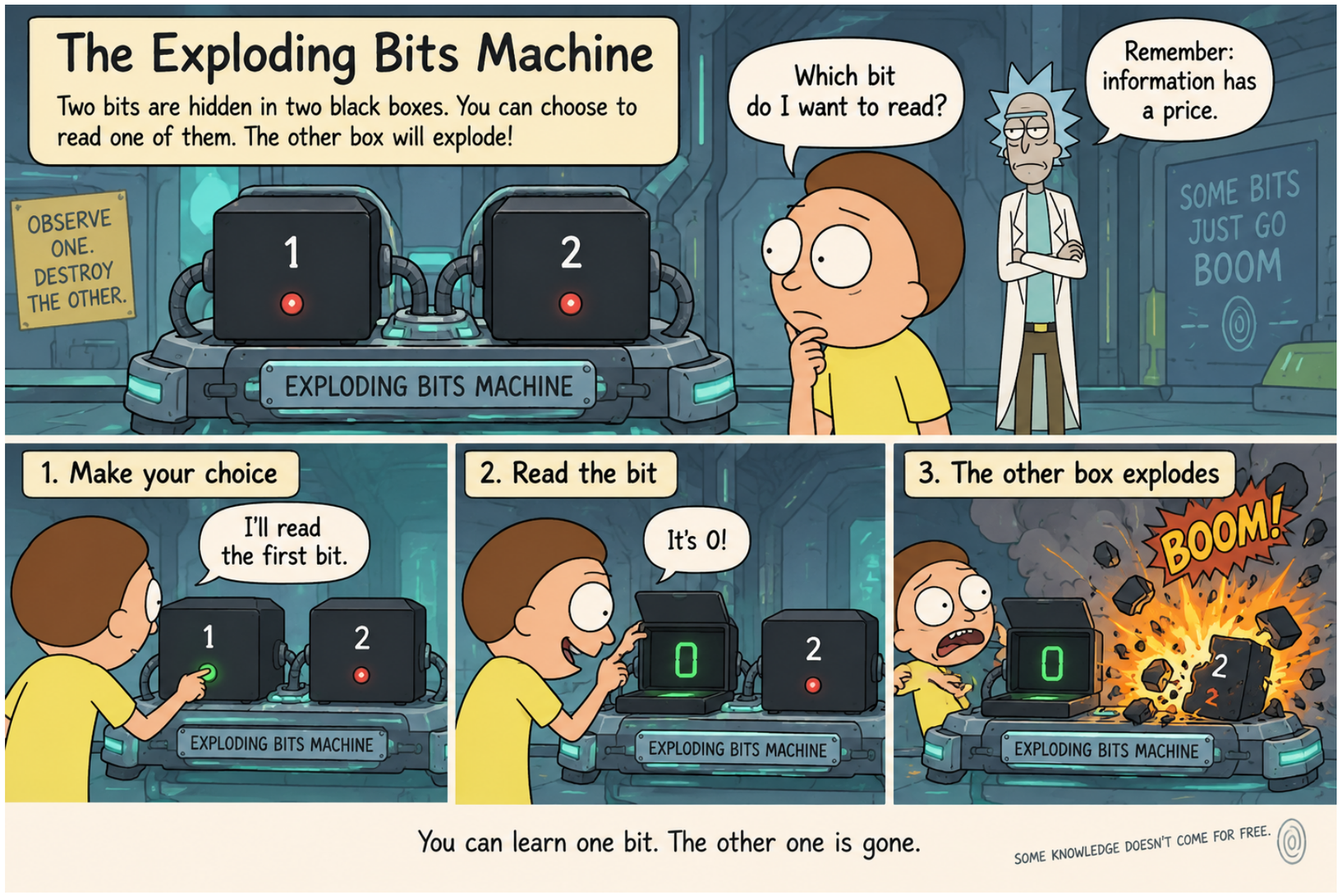
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Then $\frac{1}{2}P_{0,0} + \frac{1}{2}P_{1,1}$

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are operationally equivalent, represented by the same **state**.

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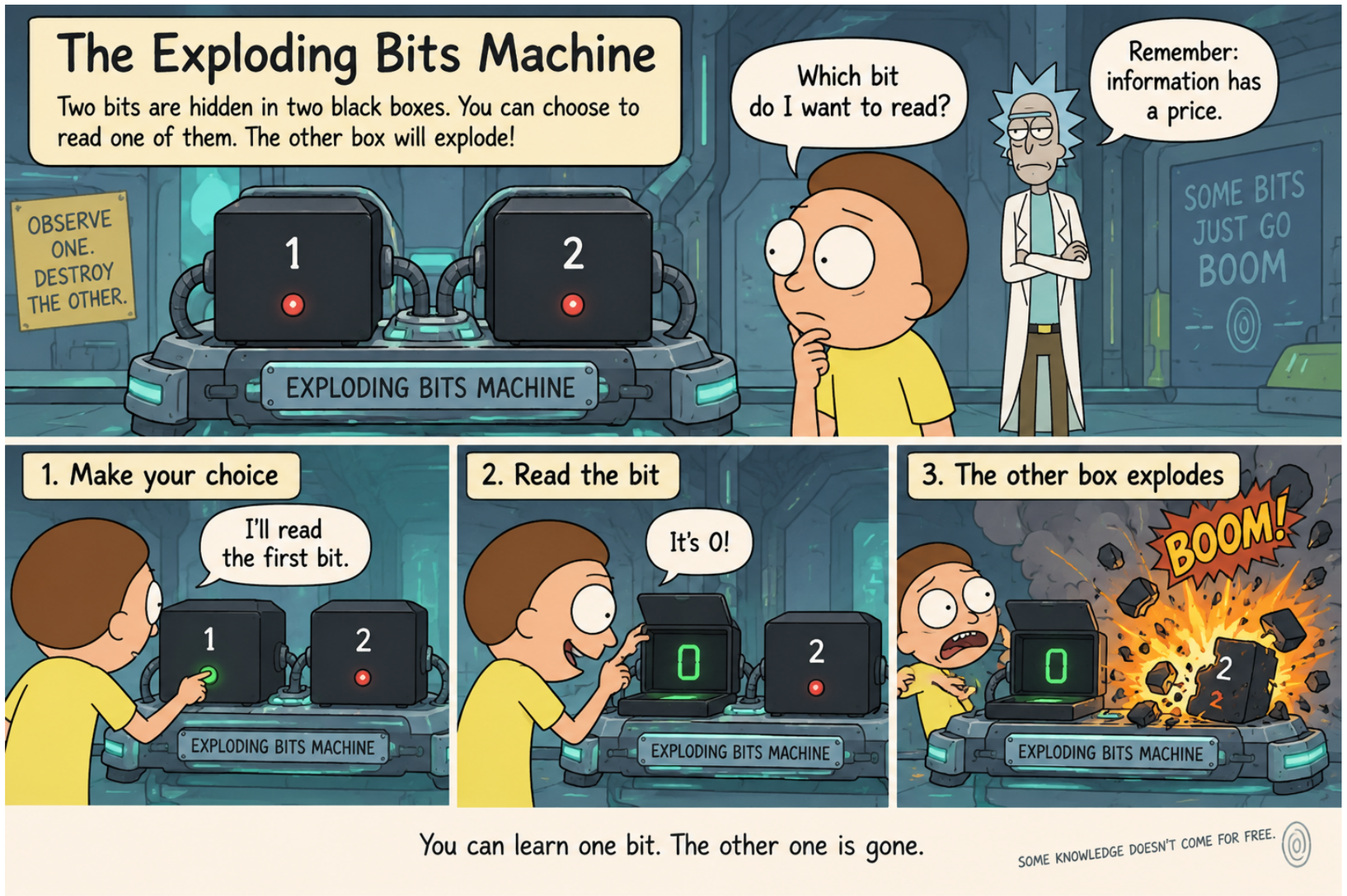
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Call the two bits *x* and *z*, then every state is determined by two parameters:

$$p_1 = \mathrm{Prob}(x \text{ is observed to be } 1),$$
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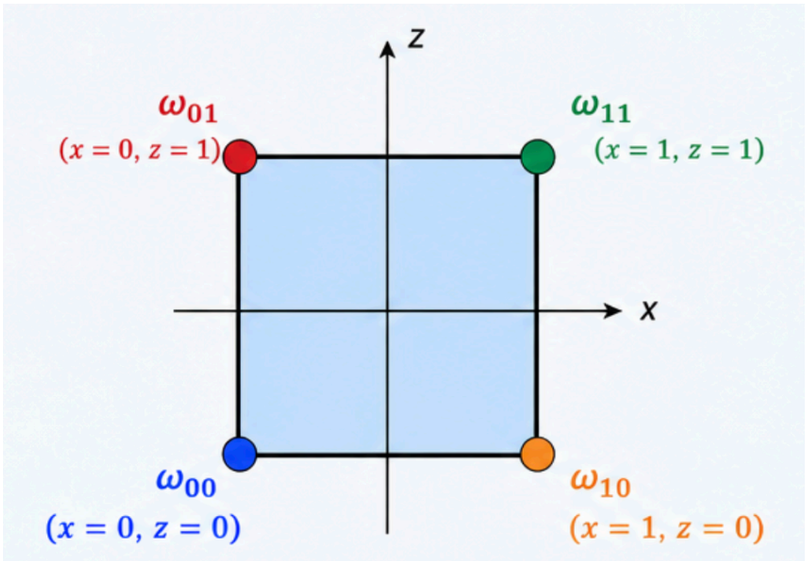
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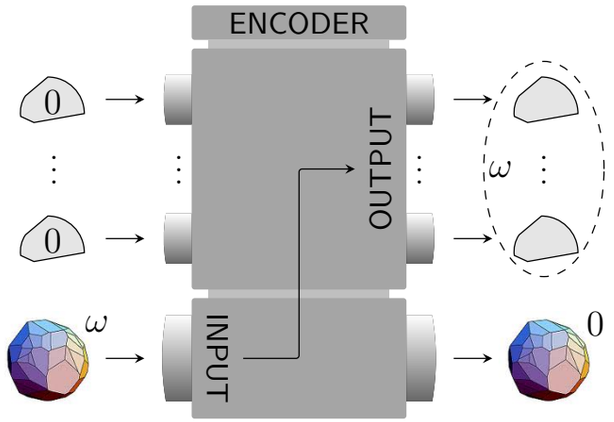
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Overview

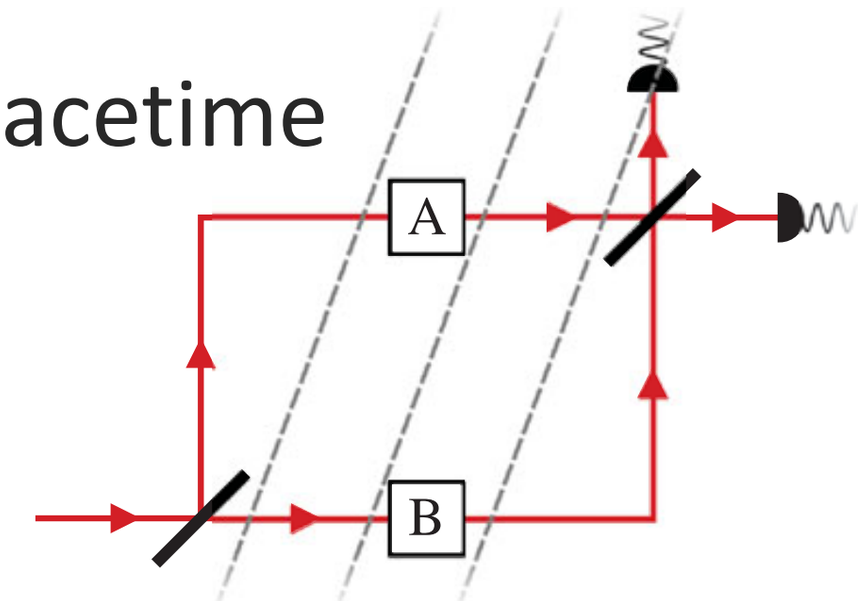
1. An (incomplete and physicists') historical perspective

2. Generalized probabilistic theories



3. An example reconstruction of QT

4. Beyond reconstructions: QT and spacetime

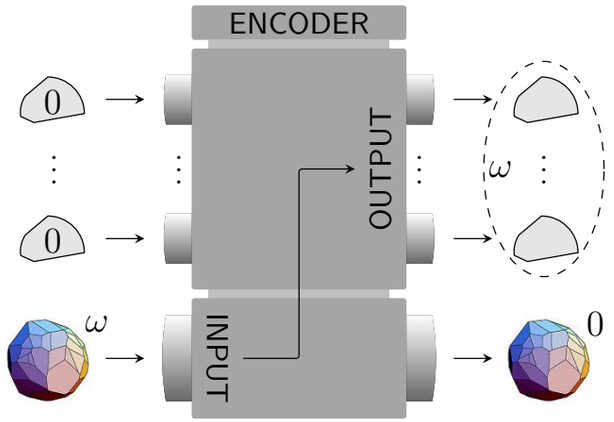


5. Conclusion

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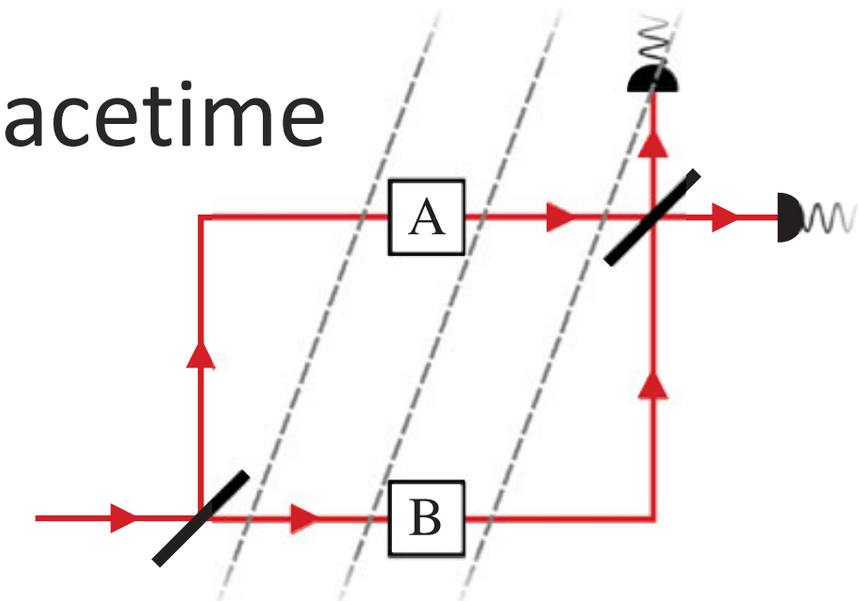
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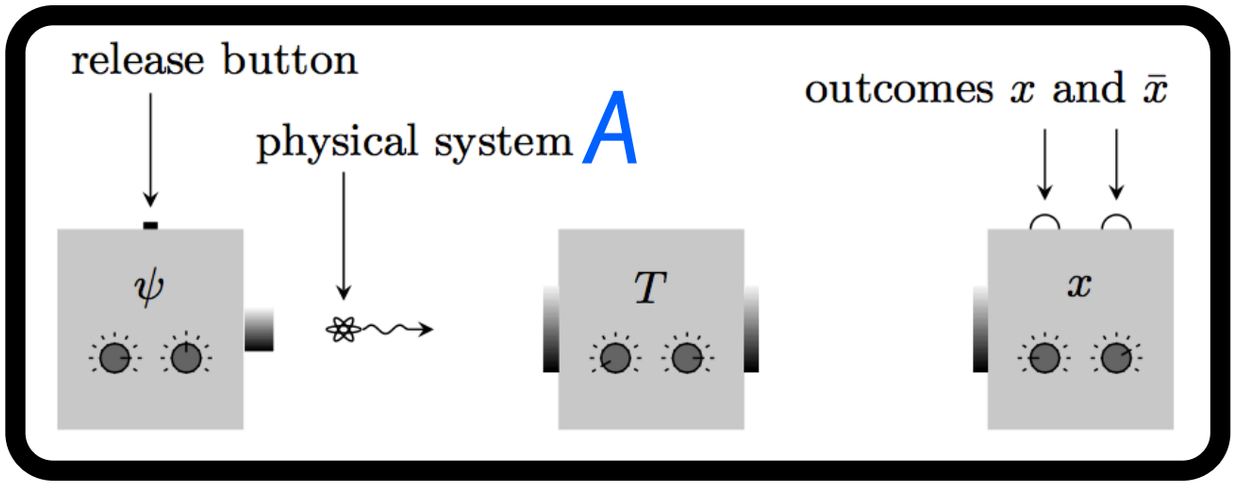
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4. Beyond reconstructions: QT and spacetime

5. Conclusion



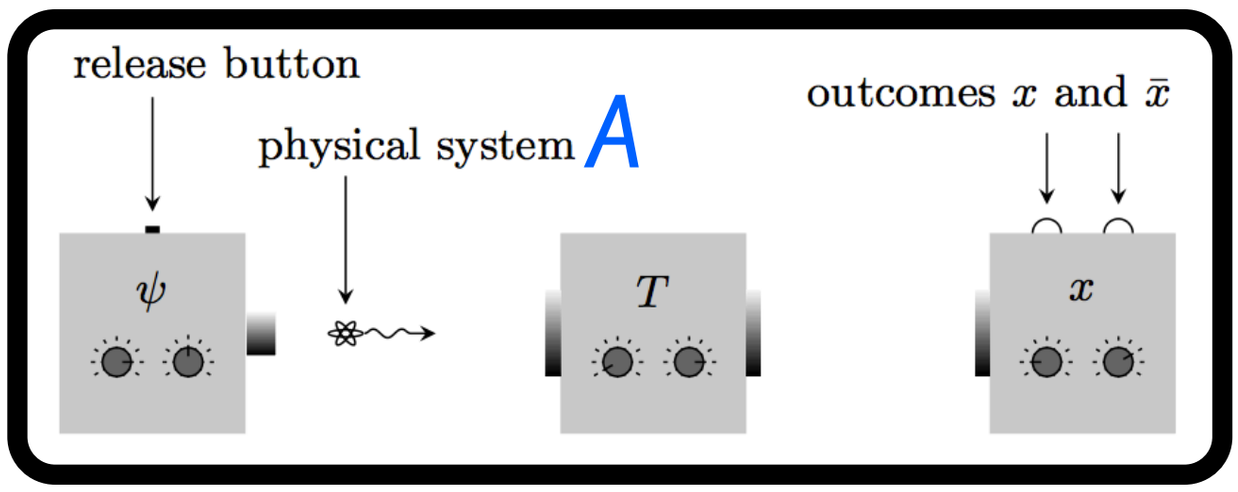
QT from three principles ("axioms")



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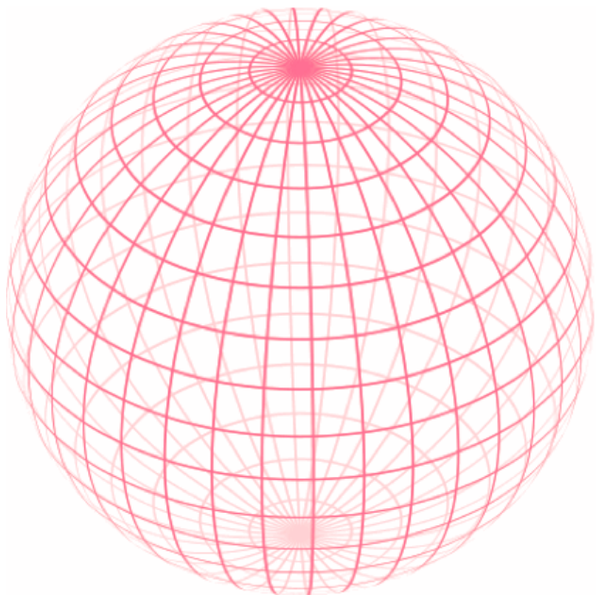
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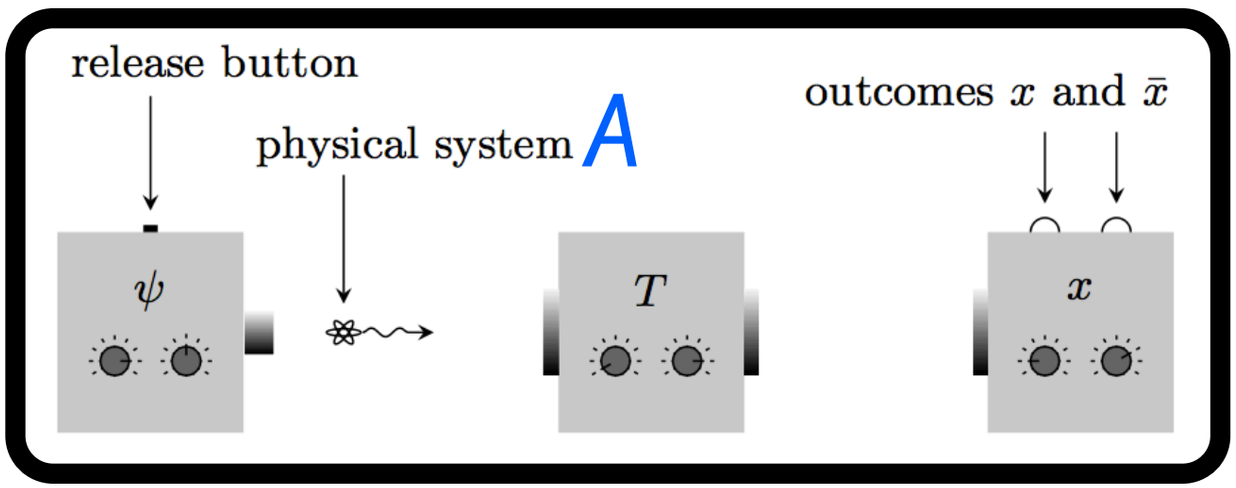
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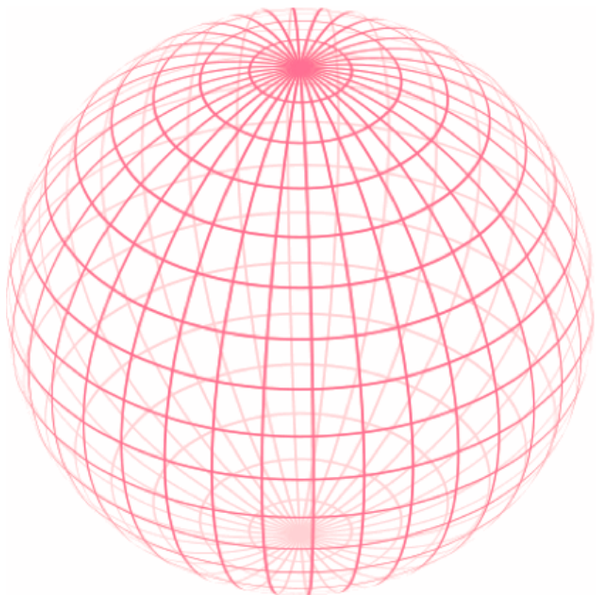
✓

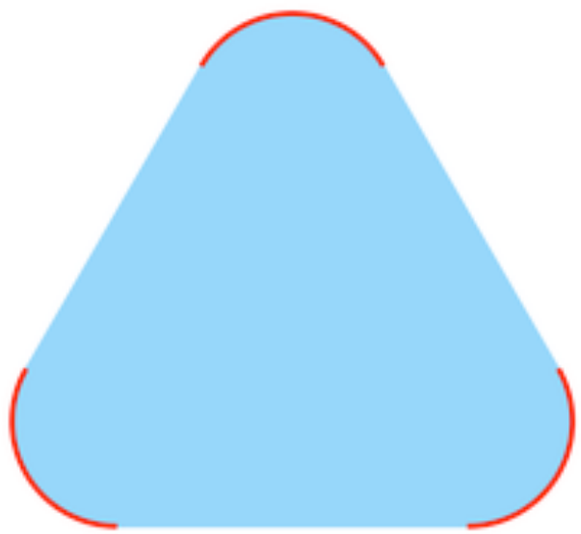
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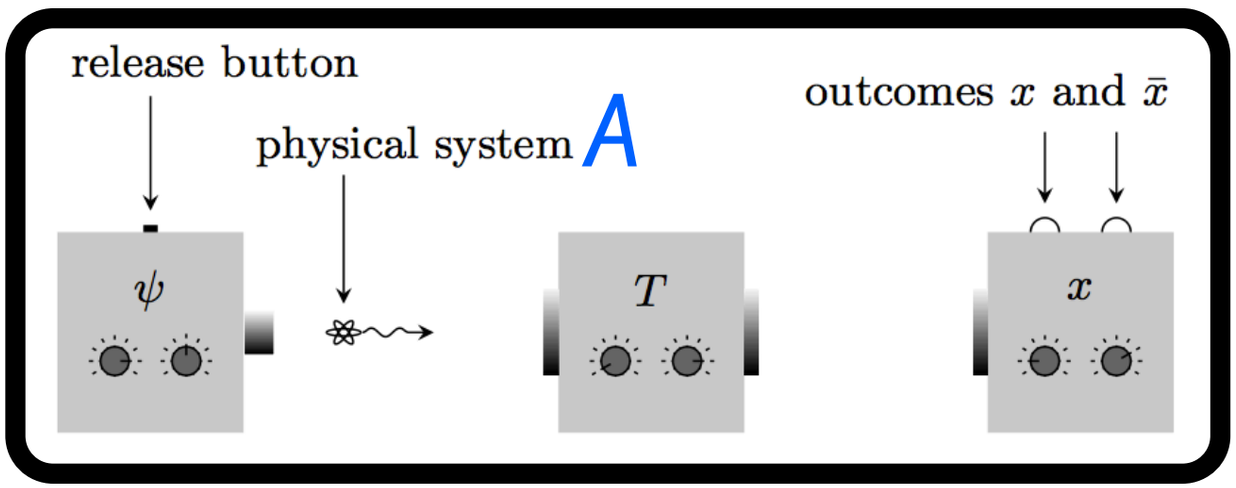
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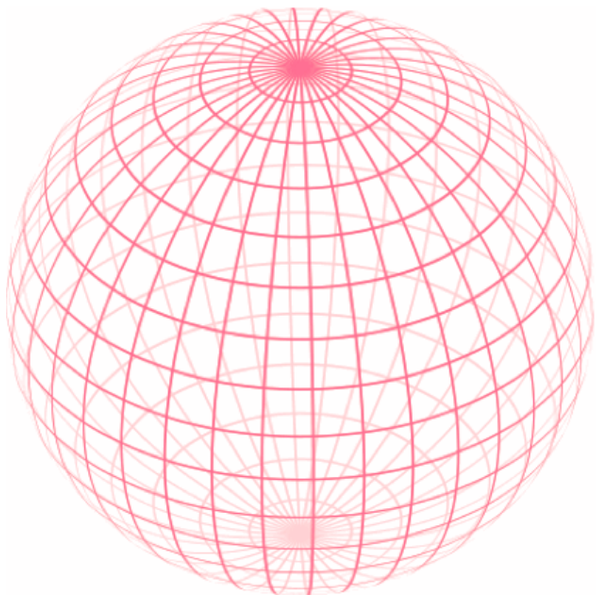


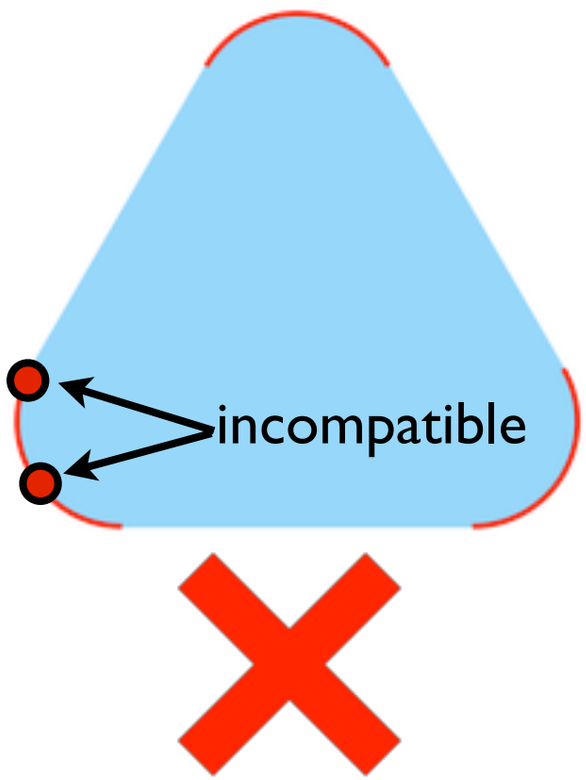
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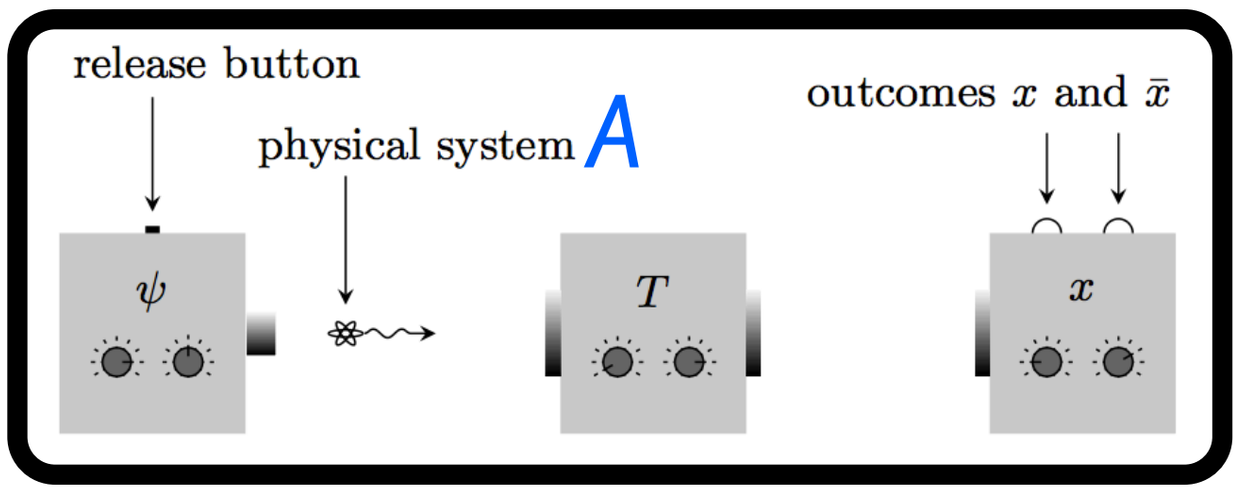
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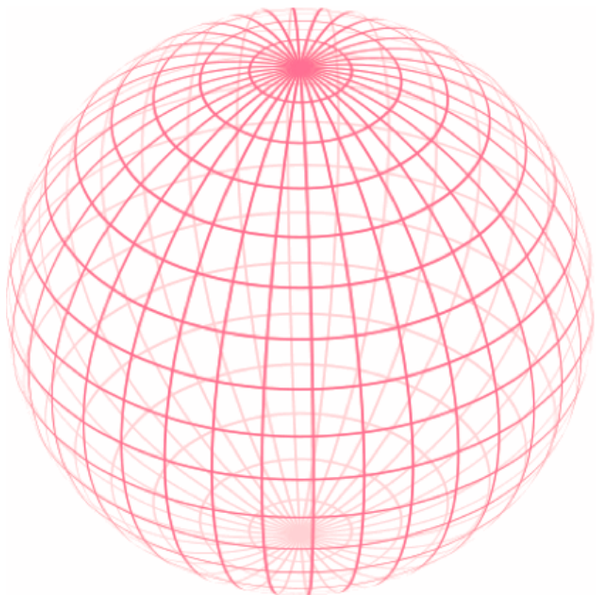


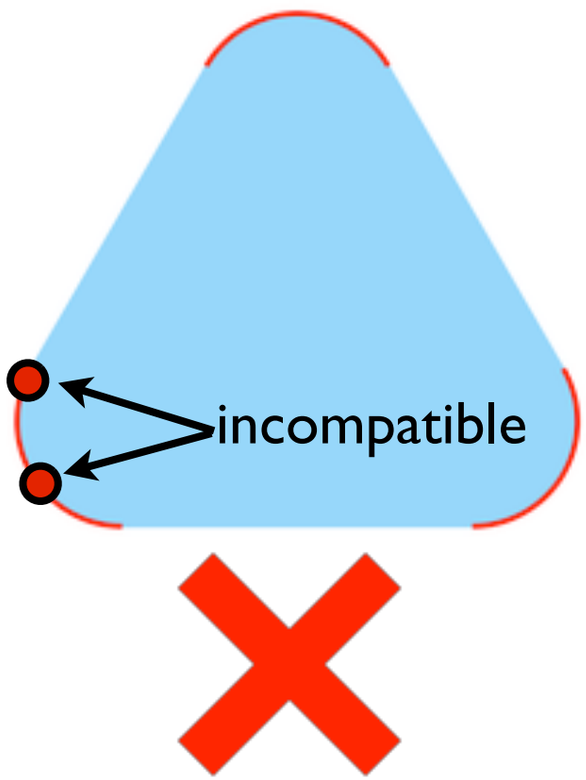
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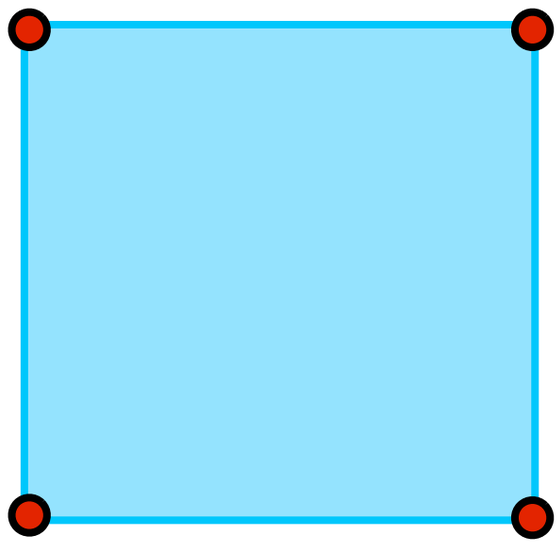


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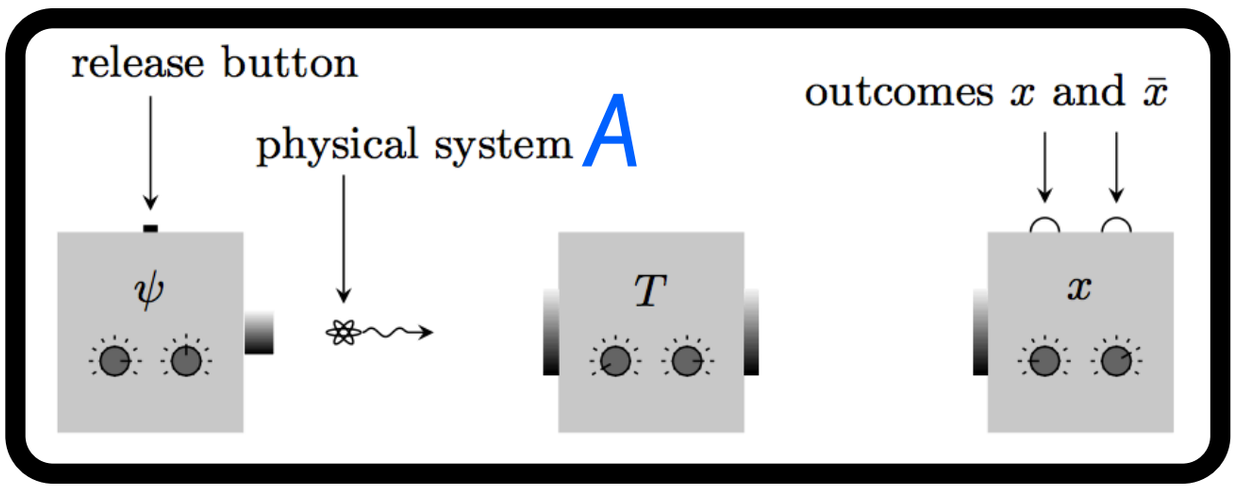
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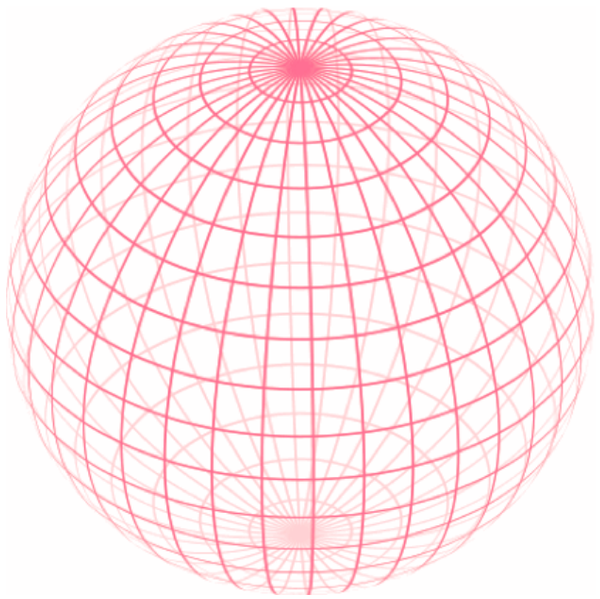


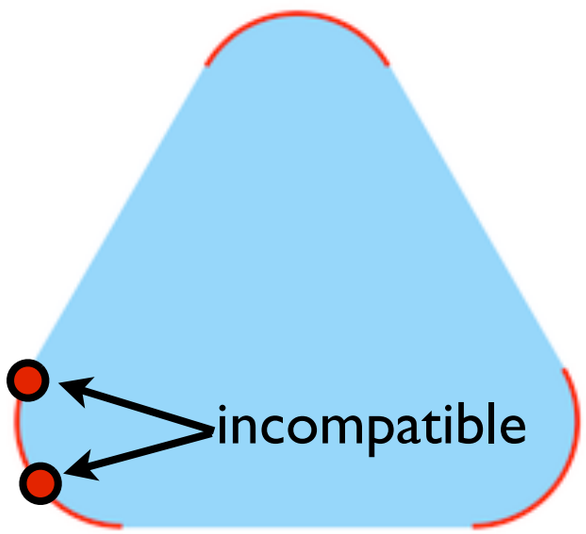
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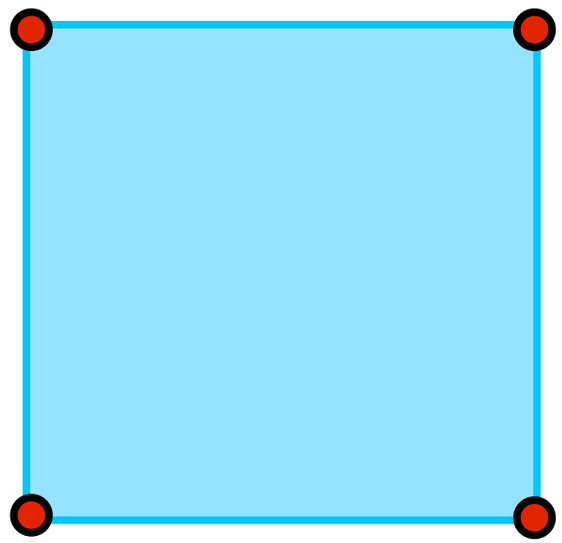


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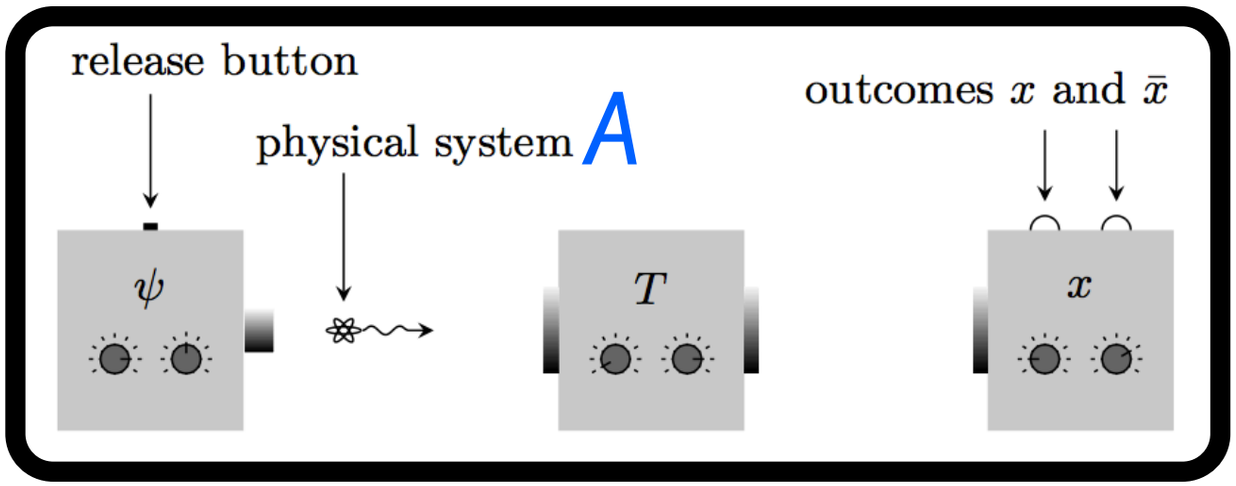
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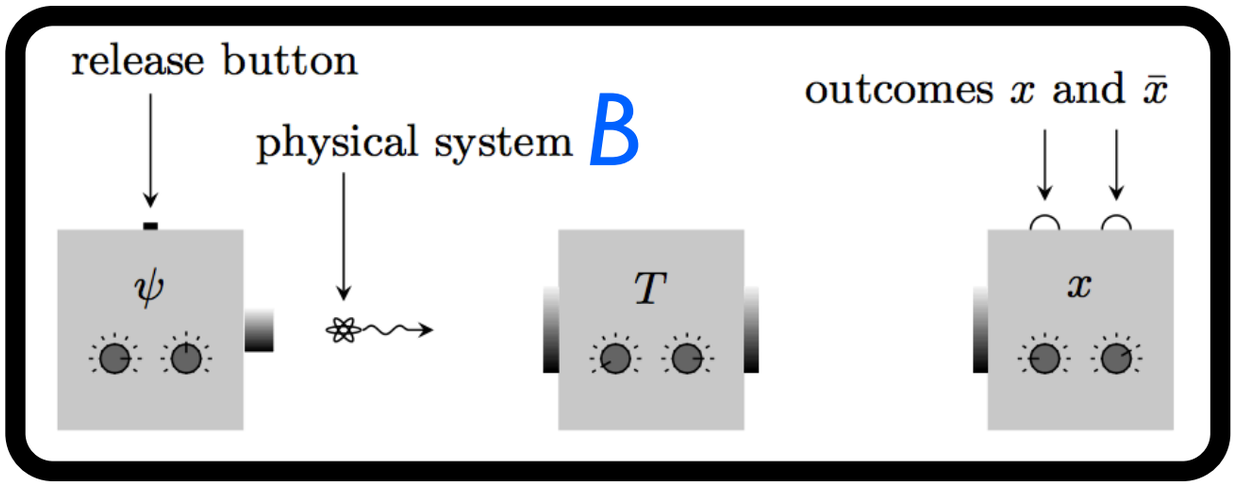




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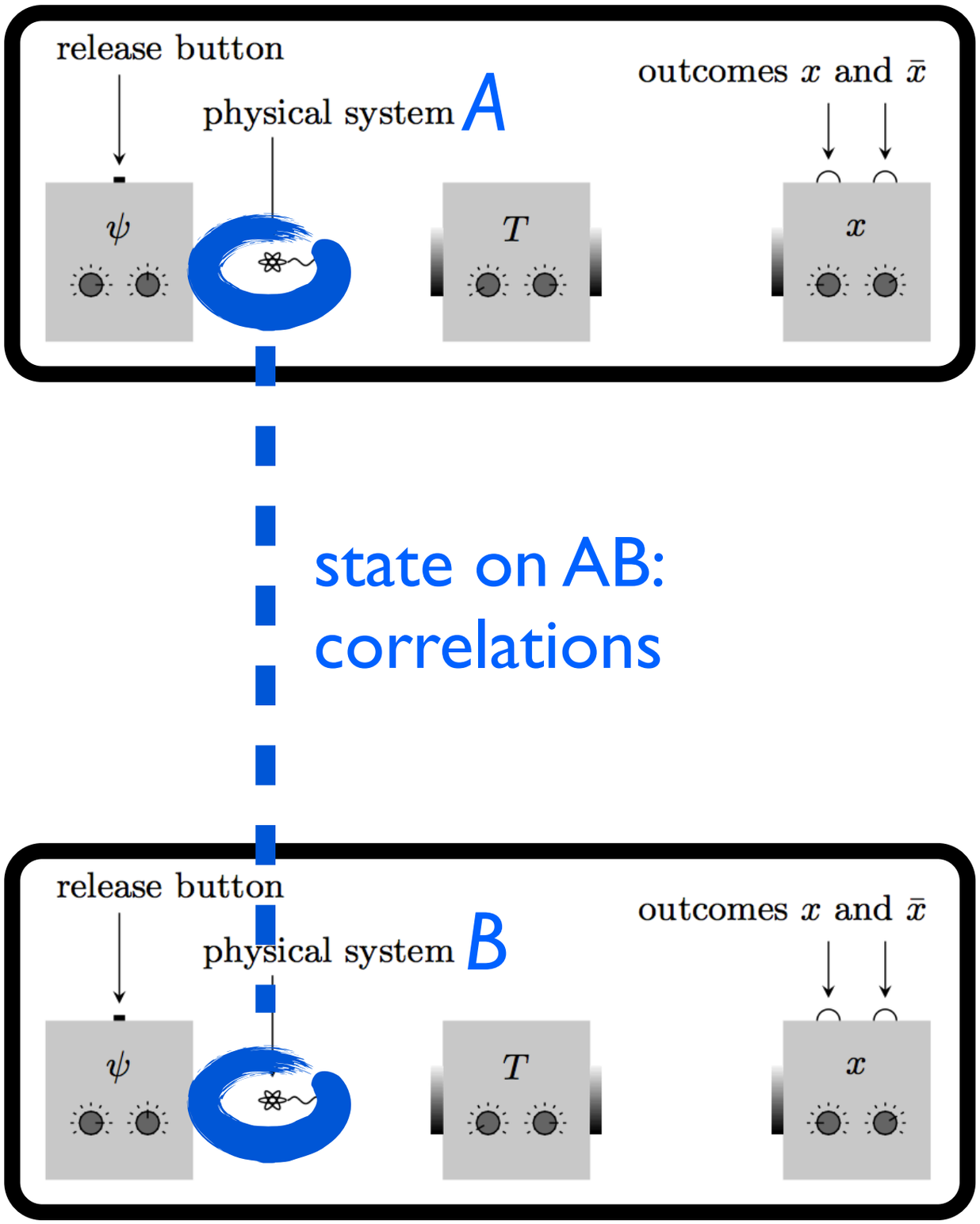








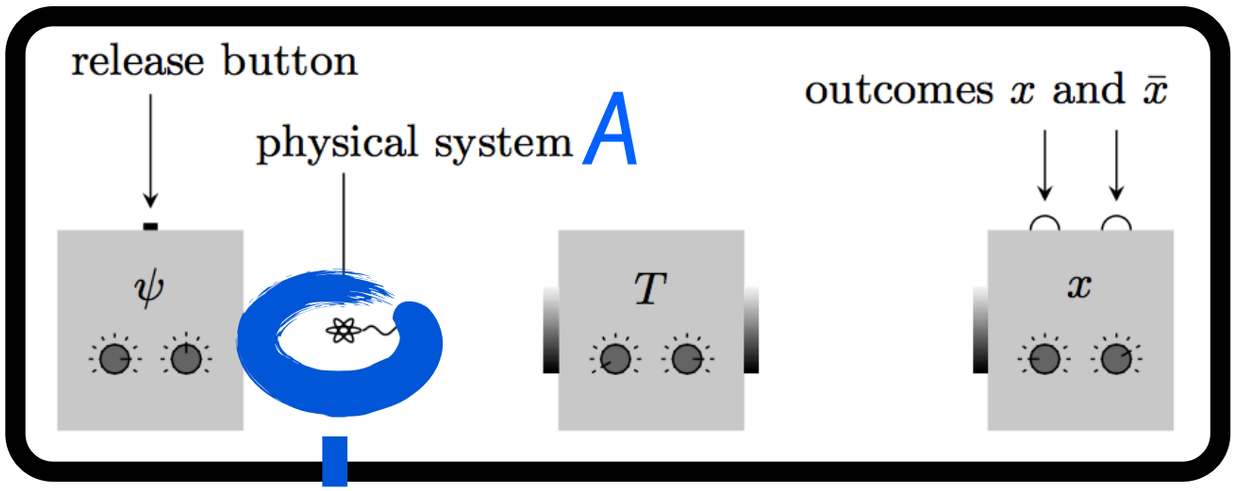
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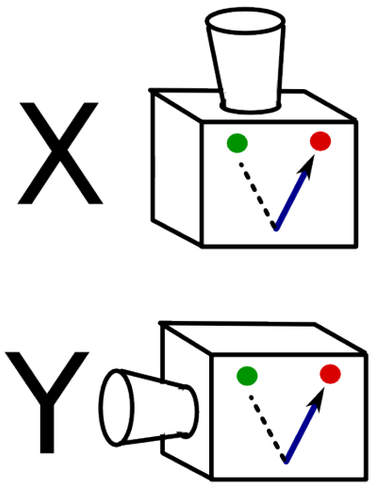




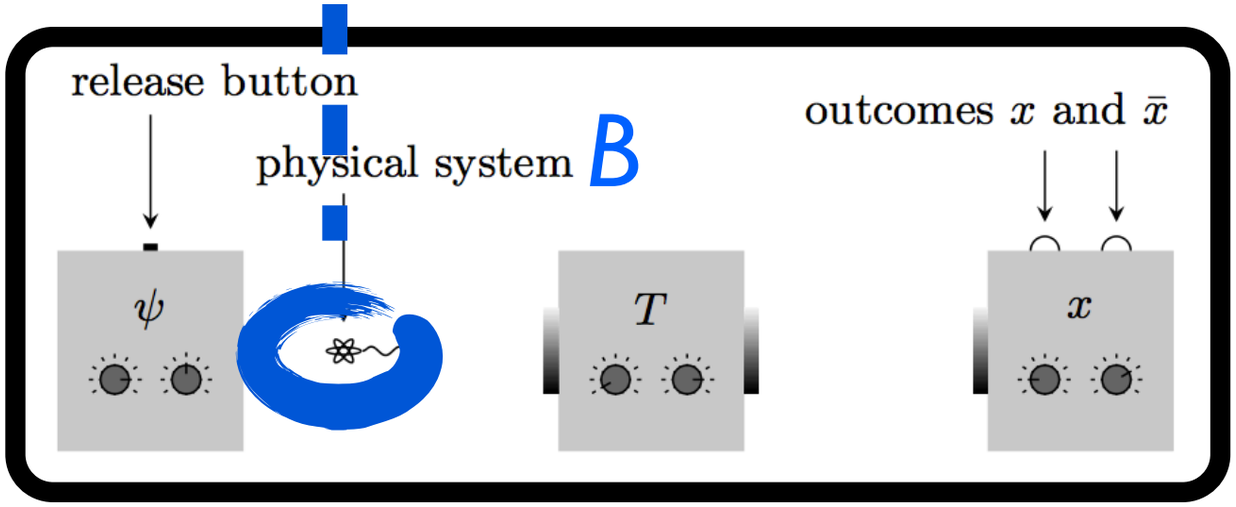
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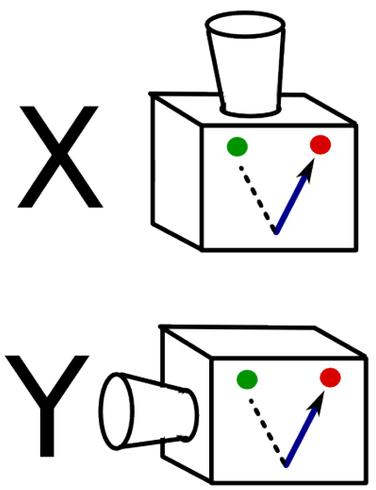




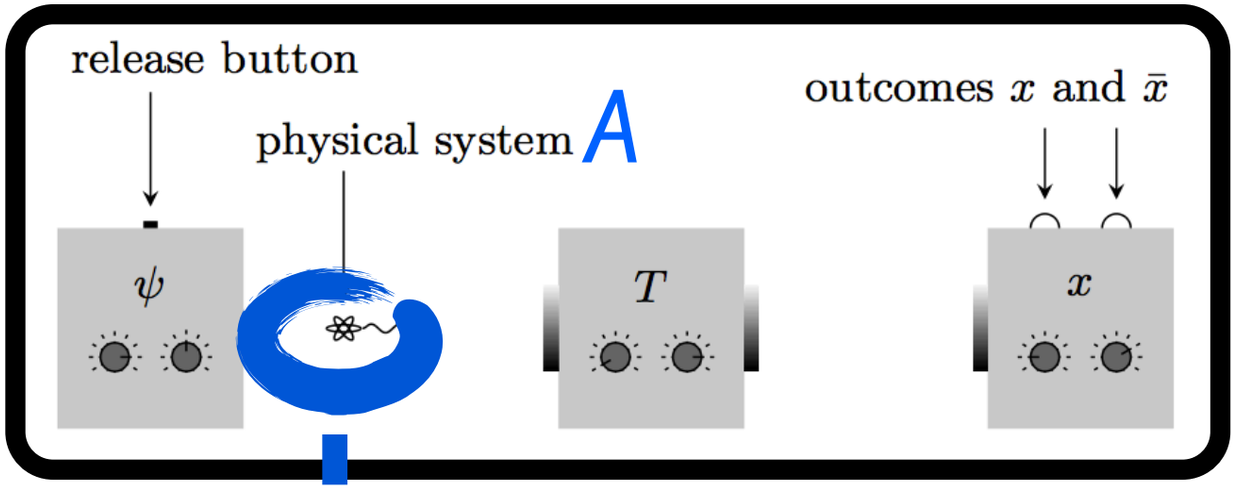
state on AB:
correlations



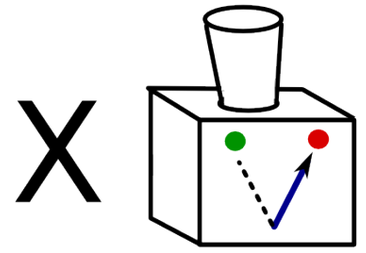




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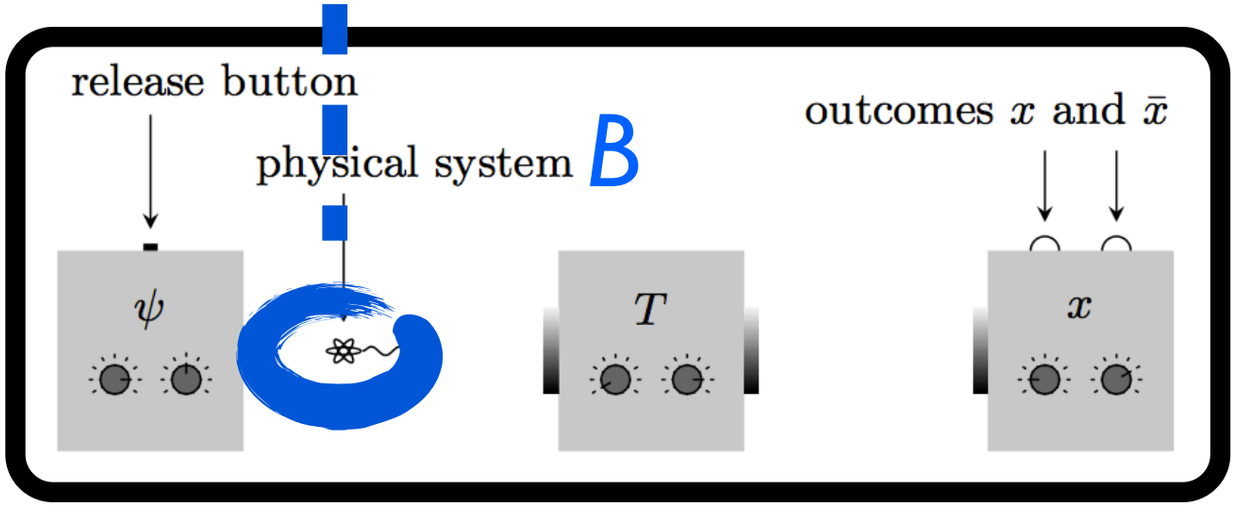




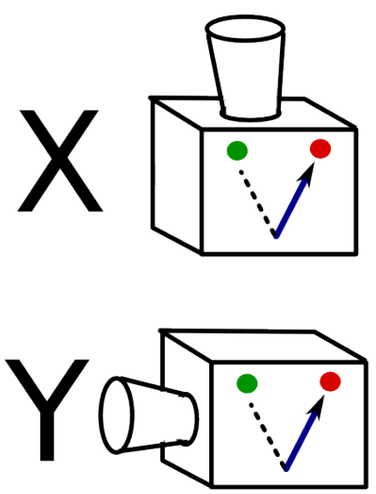


state on AB: correlations

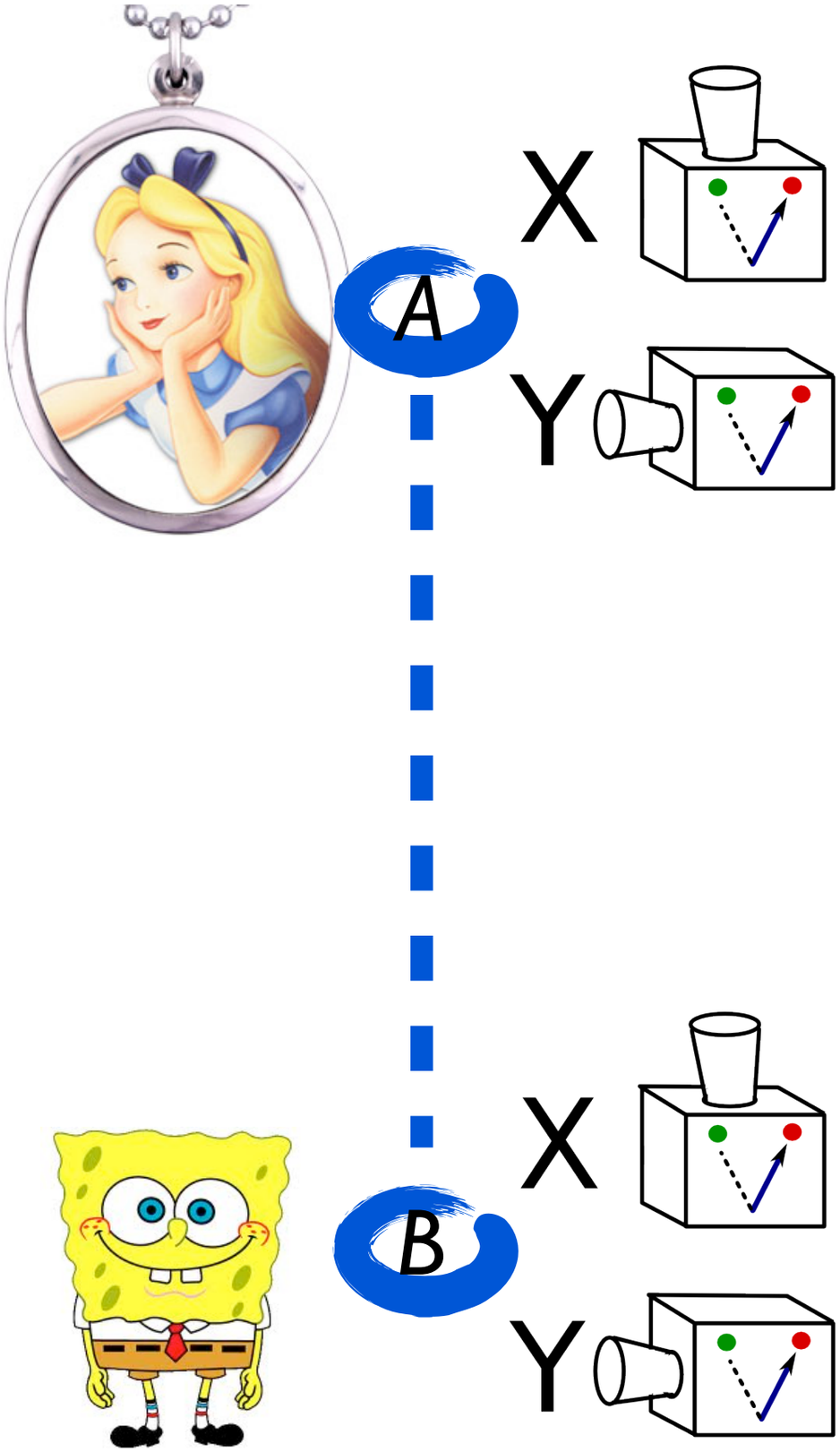
No-signalling condition:
Alice‘s probabilities do not depend on Bob‘s choice of measurement.



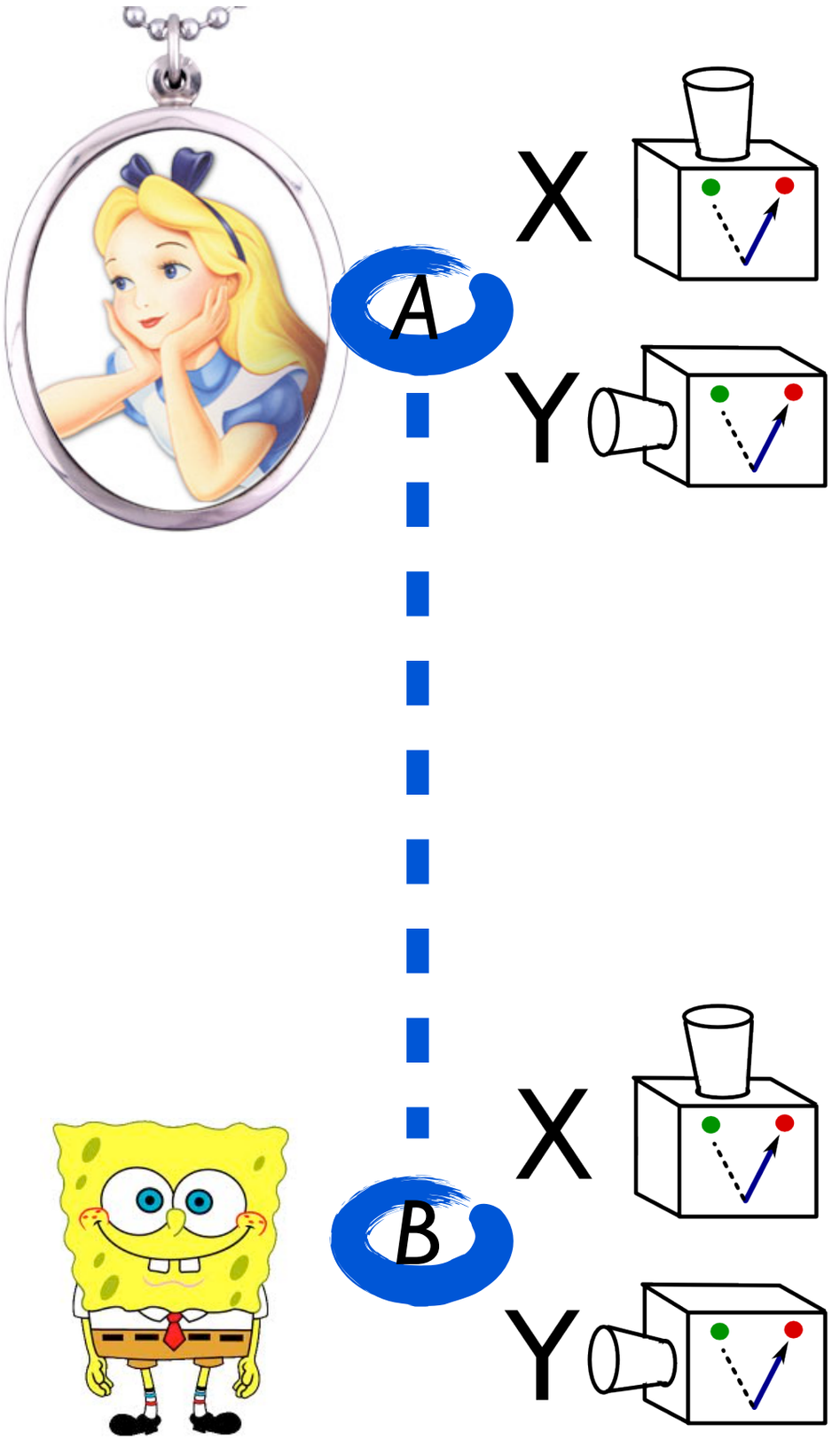




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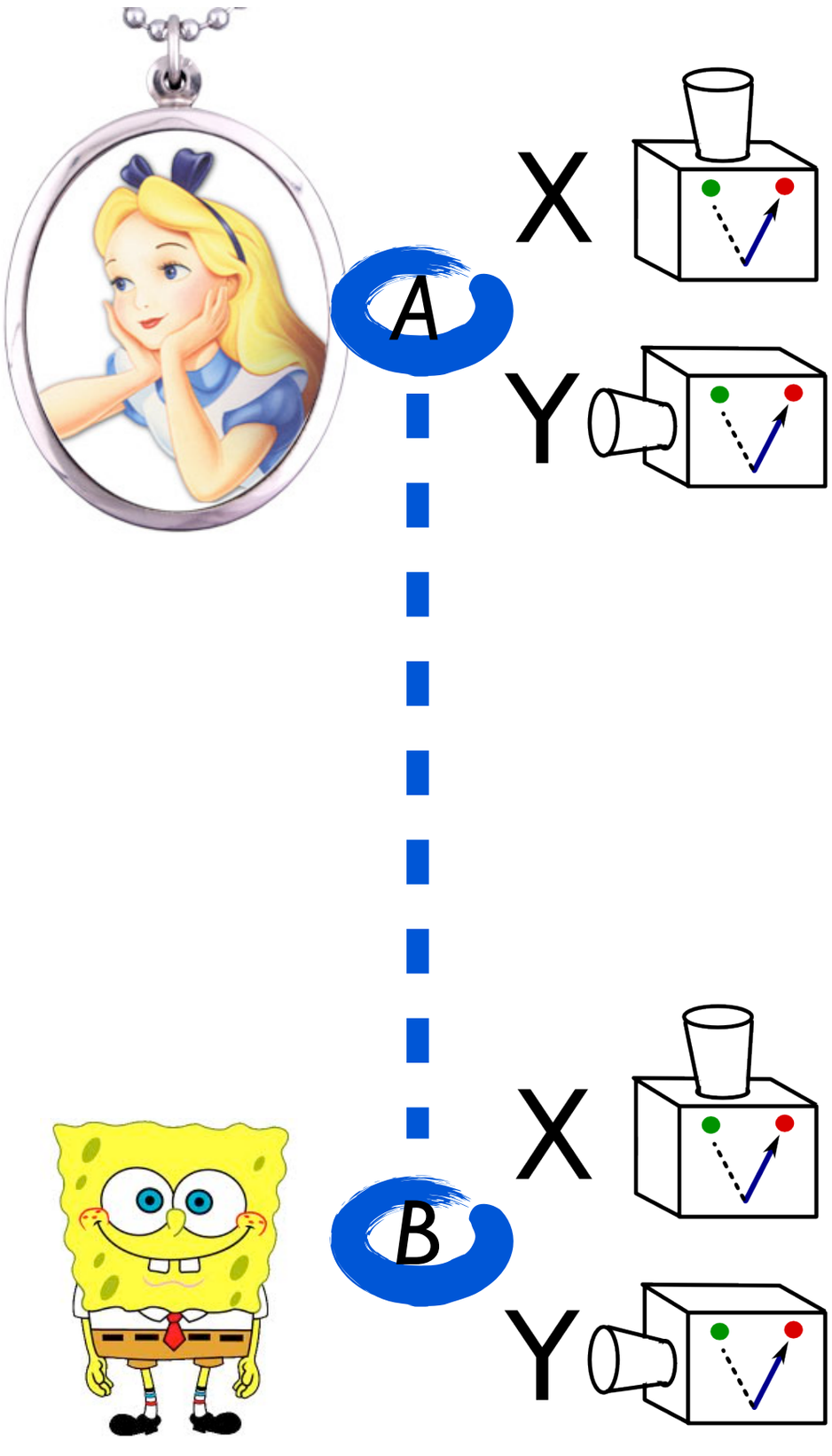


QT from three principles (“axioms”)



Axiom II: States on AB are uniquely determined by correlations of local measurements on A,B.

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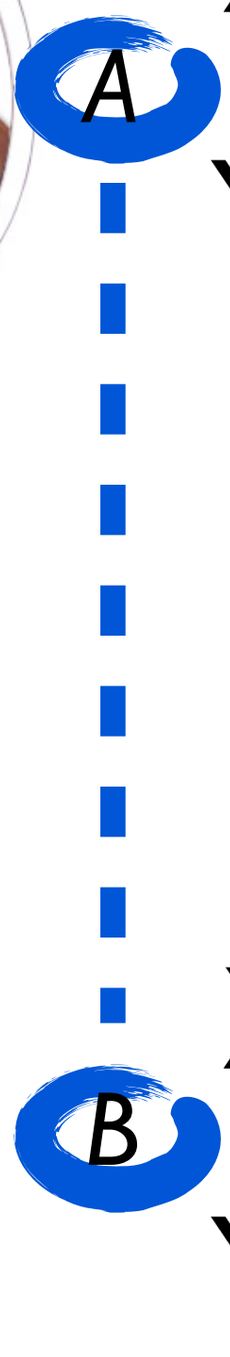


Axiom II: States on AB are uniquely determined by correlations of local measurements on A,B.

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No non-local measurements necessary.

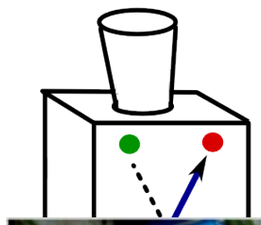
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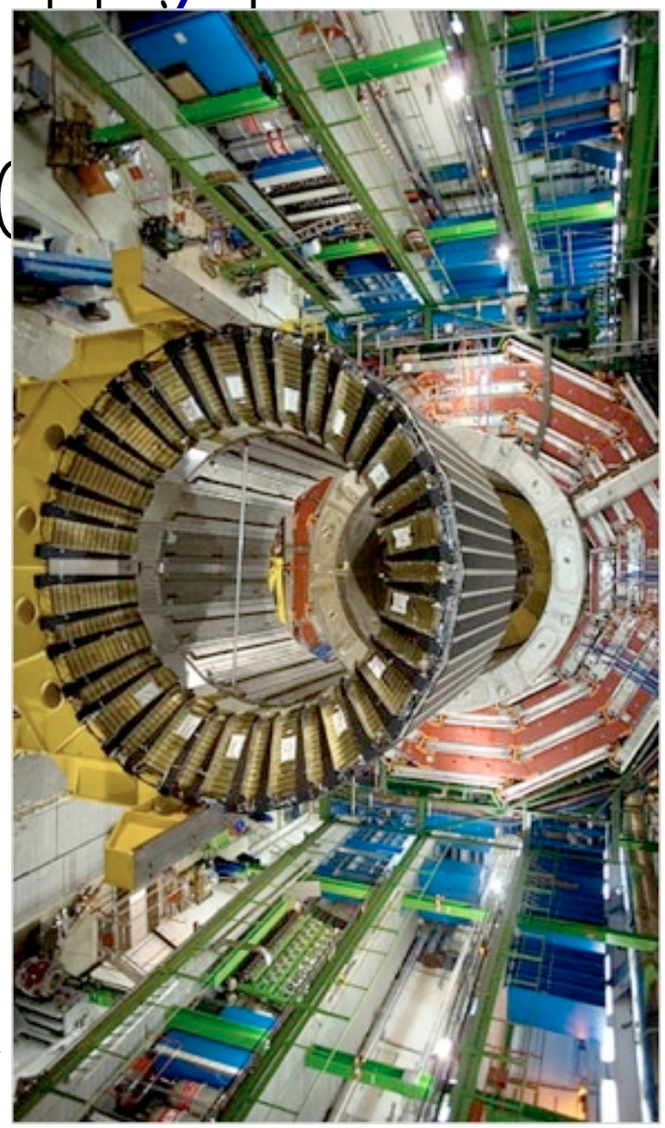




X

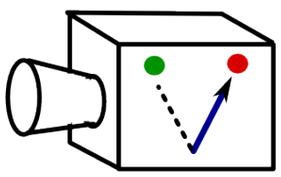
Y





X

Y

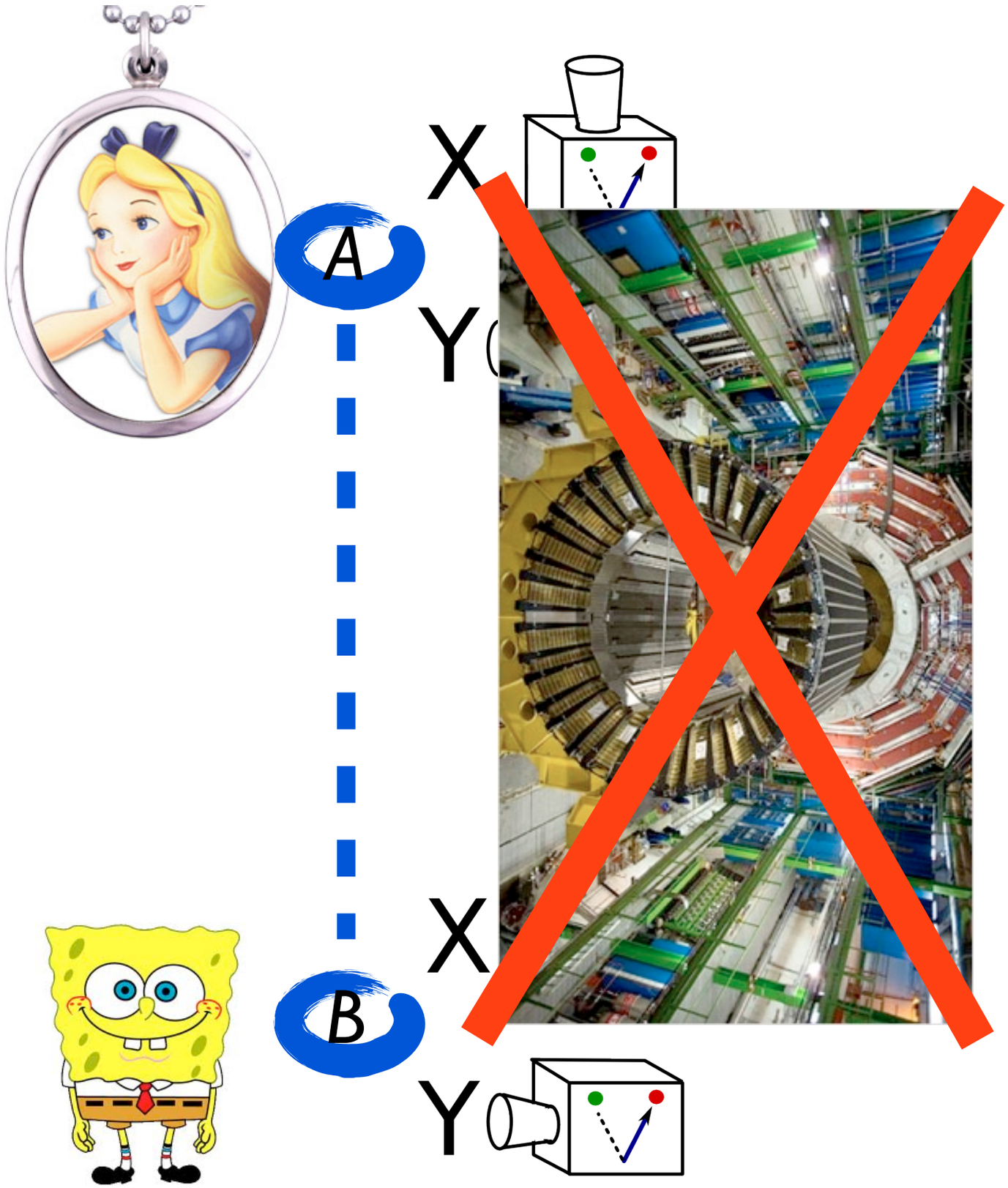




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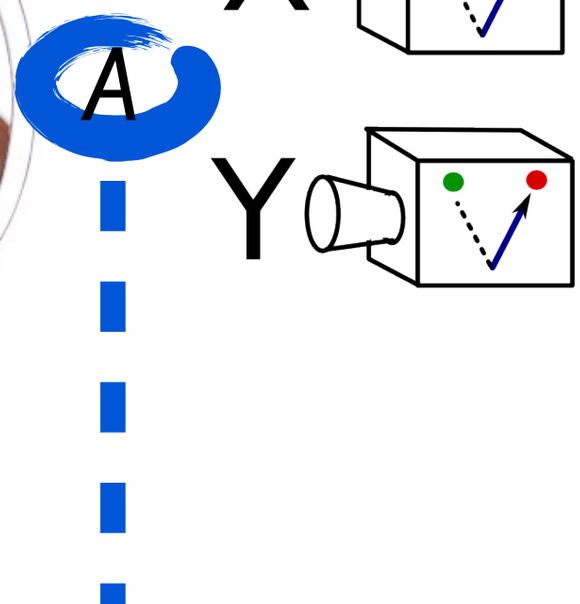


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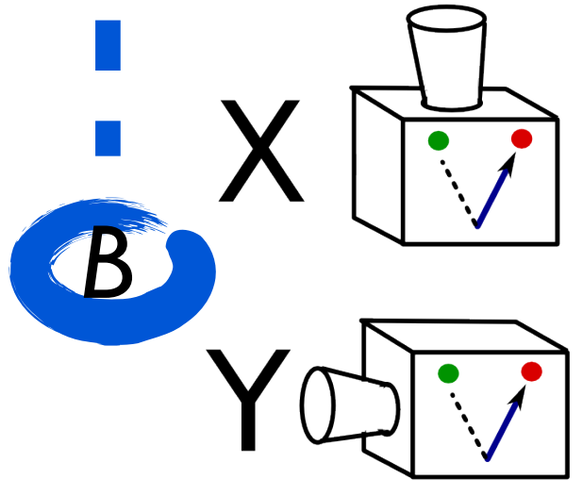




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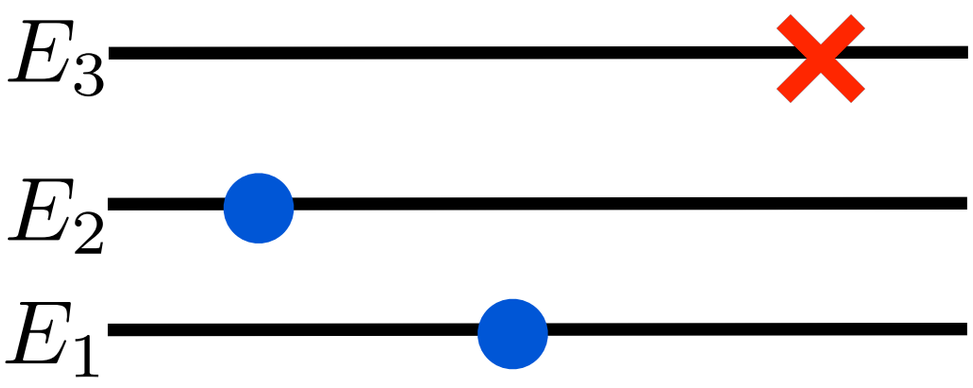




Global state space $\Omega_{AB} \subset A \otimes B$
but not uniquely fixed!

Axiom III: The Subspace Axiom

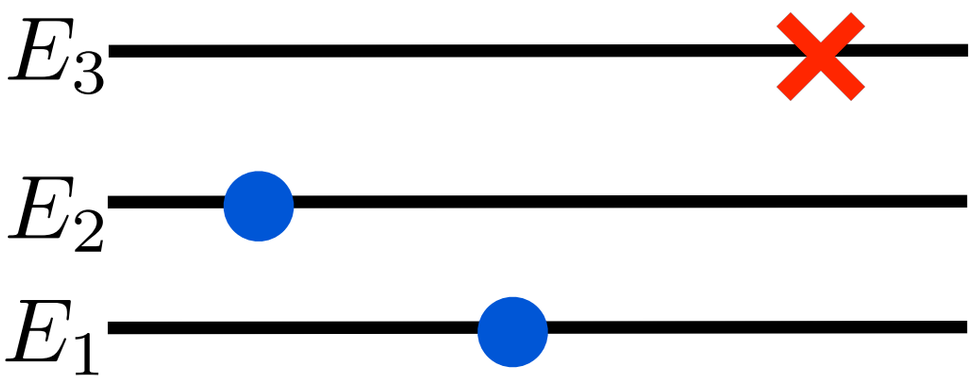
Some 3-level system:



Impossible to have system in 3rd level
⇒ find particle there with probab. 0

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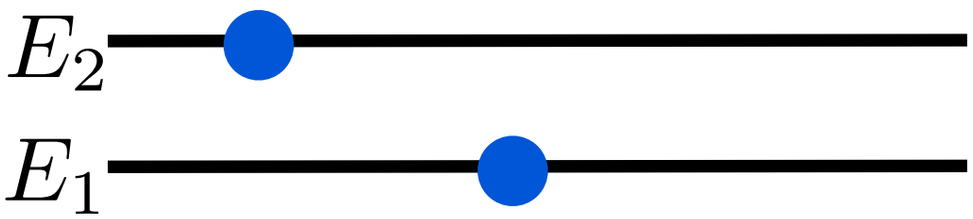
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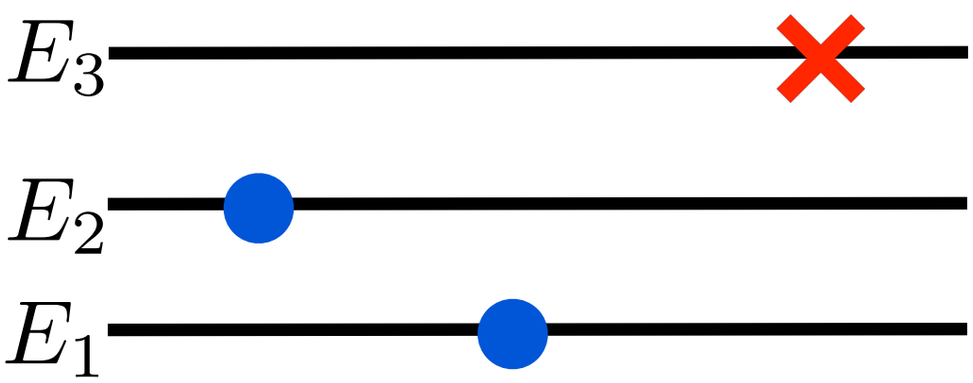
=



2-level system.

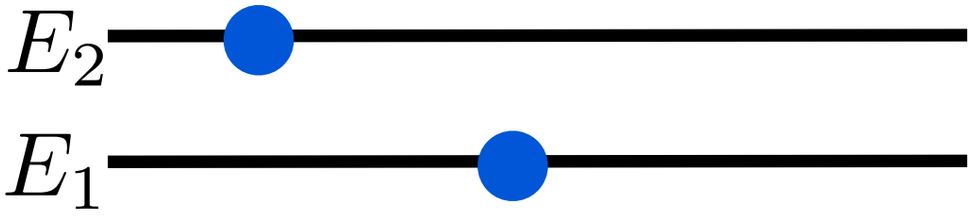
Axiom III: The Subspace Axiom

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Impossible to have system in 3rd level
$\Rightarrow$ find particle there with probab. 0





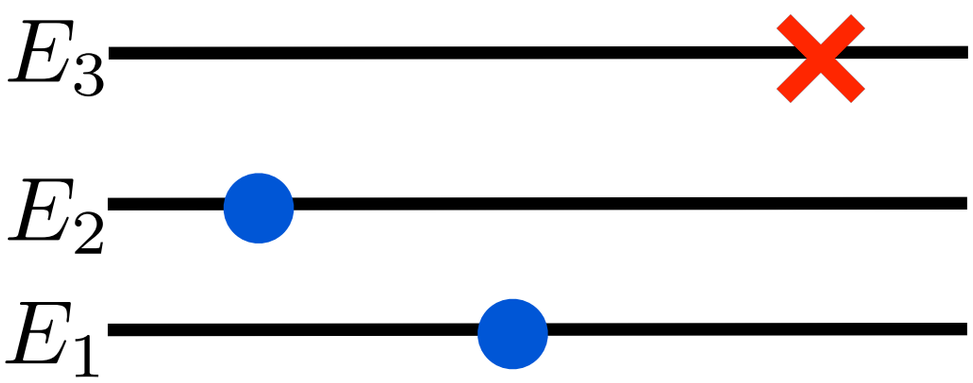
2-level system.

QT: $\rho^{(3)} = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \rho^{(2)} = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$

CPT: $P^{(3)} = (P_1, P_2, 0) \longrightarrow P^{(2)} = (P_1, P_2)$

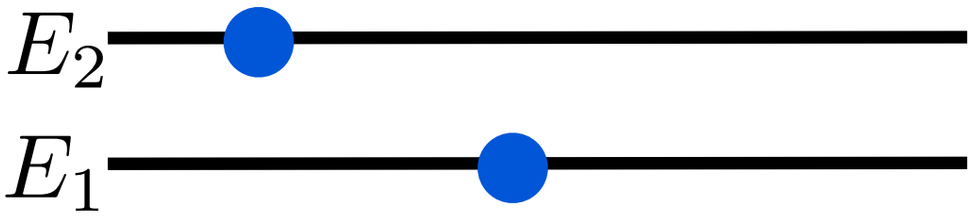
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Otherwise, physics would be affected by impossible counterfactuals.

Axiom III: The Subspace Axiom

Axiom III: Let Ω_N and Ω_{N-1} be systems with capacities *N* and *N-1*. If $(E_1, \ldots, E_N)$ is a complete measurement on Ω_N, then the set of states ω with $E_N(\omega) = 0$ is equivalent to Ω_{N-1}.

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Capacity N of Ω = maximal # of perfectly distinguishable states.

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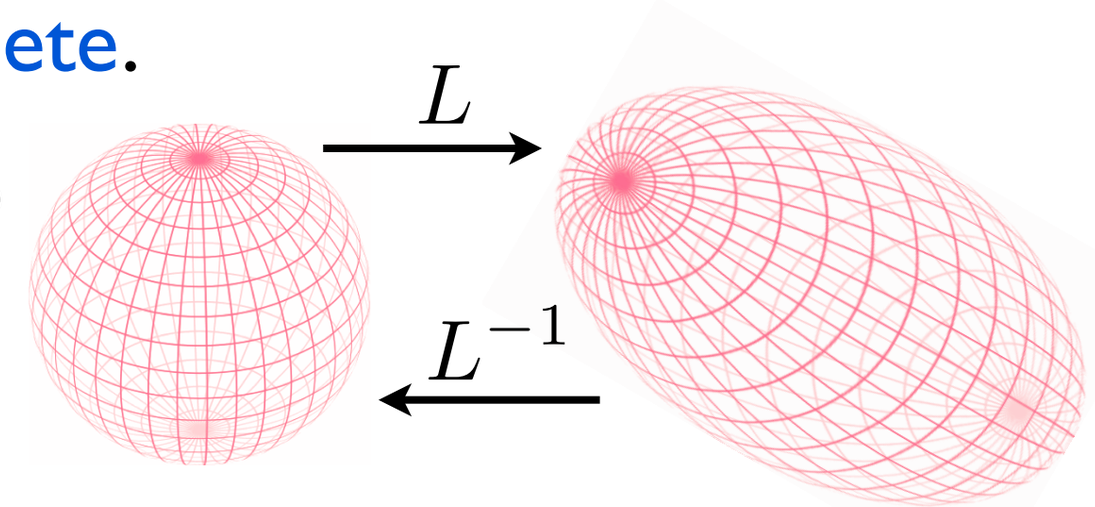
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Equivalent = same state spaces up to a linear map (physically the same!)



Result

Ll. Masanes and M. P. Müller, *A derivation of quantum theory from physical requirements*, New J. Phys. **13**, 063001 (2011).

Theorem. There are exactly two GPTs satisfying the following three axioms:

I. Reversibility

II. Local Tomography

III. Subspace Axiom

These are:

- **Classical Probability Theory**

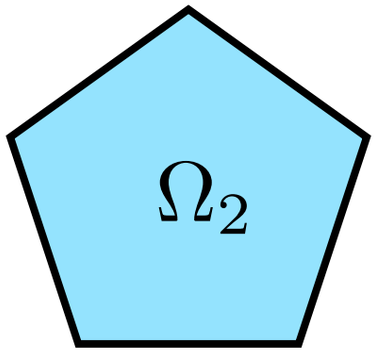
  Capacity-*n* states are the probability vectors $(p_1, \ldots, p_n)$;
  reversible transformations are permutations; the usual measurements.

- **Quantum Theory**

  Capacity-*n* states are the complex $n \times n$ density matrices;
  reversible transformations are unitary conjugations $\rho \mapsto U \rho U^\dagger$;
  the measurements are the POVMs.

Sketch of first small part of proof

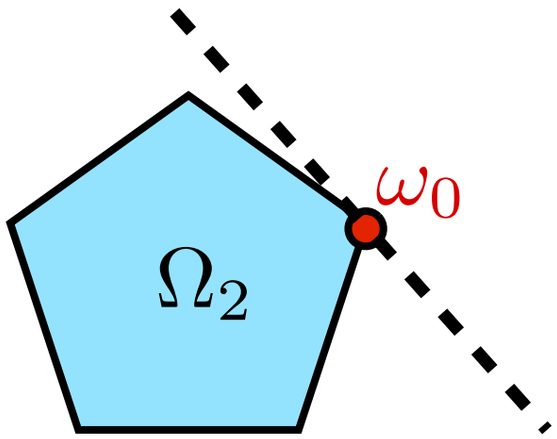
Why a bit is described by a ball:



capacity 2 (bit)

Sketch of first small part of proof

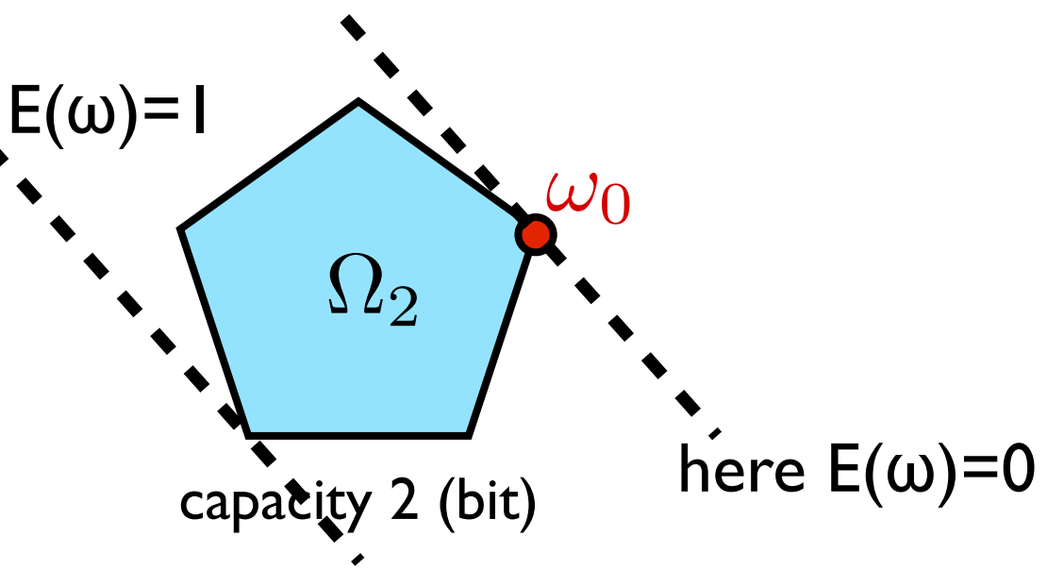
Why a bit is described by a ball:



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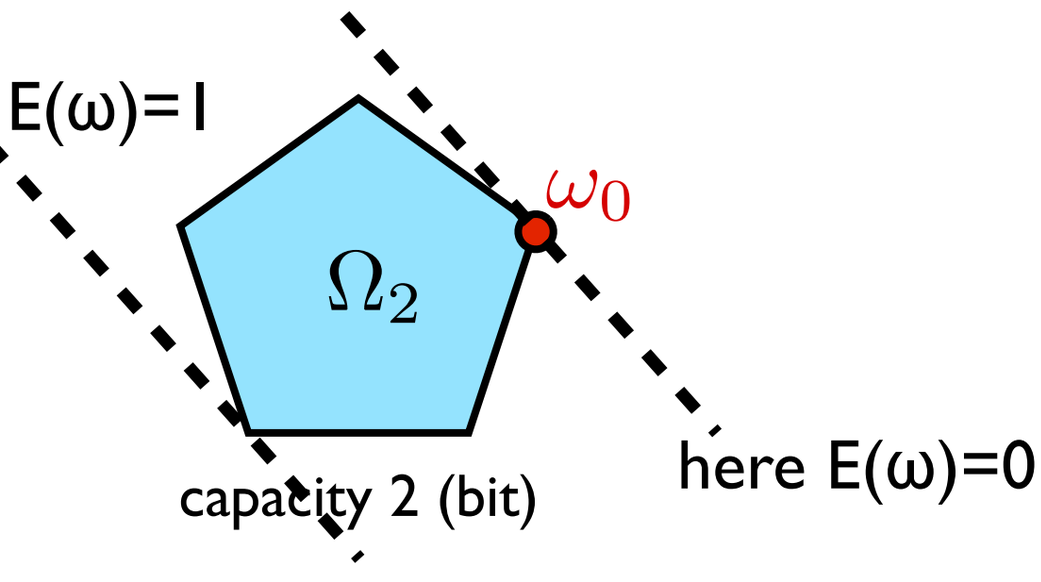


(I-E, E) is complete measurement.

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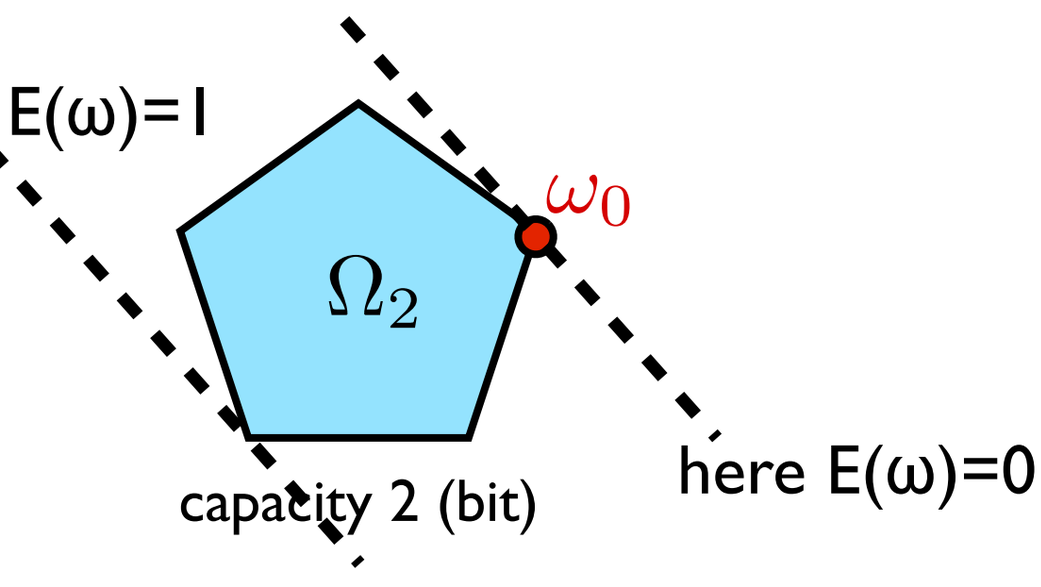
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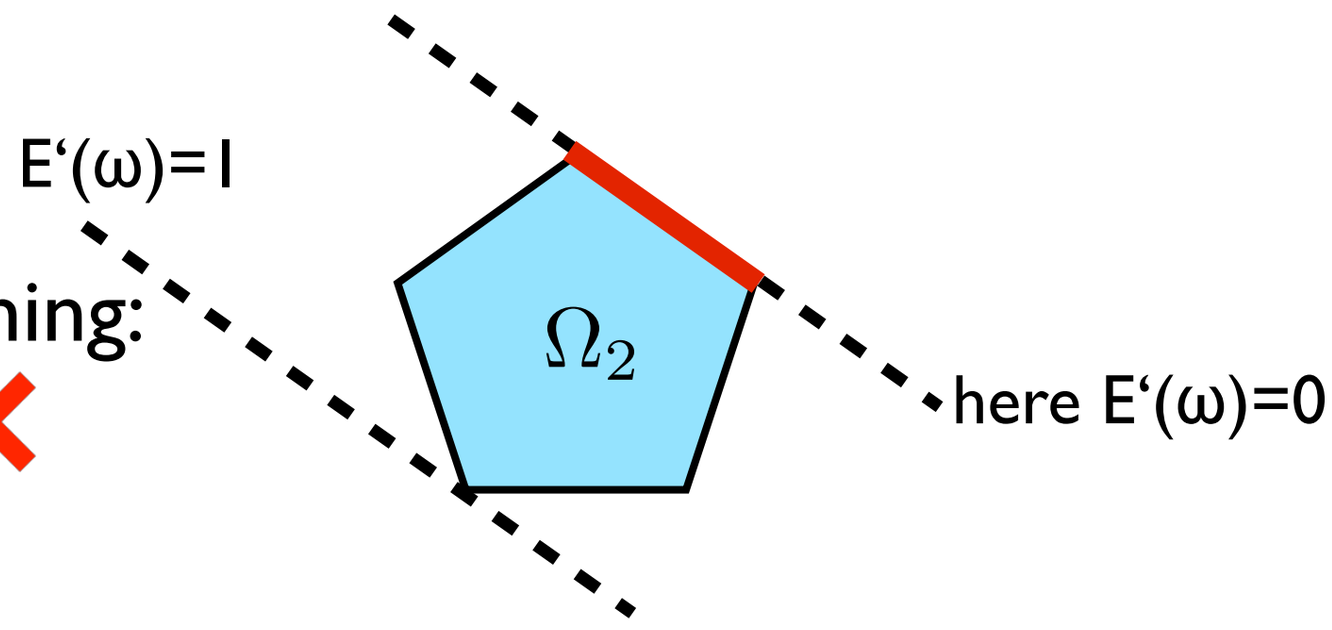
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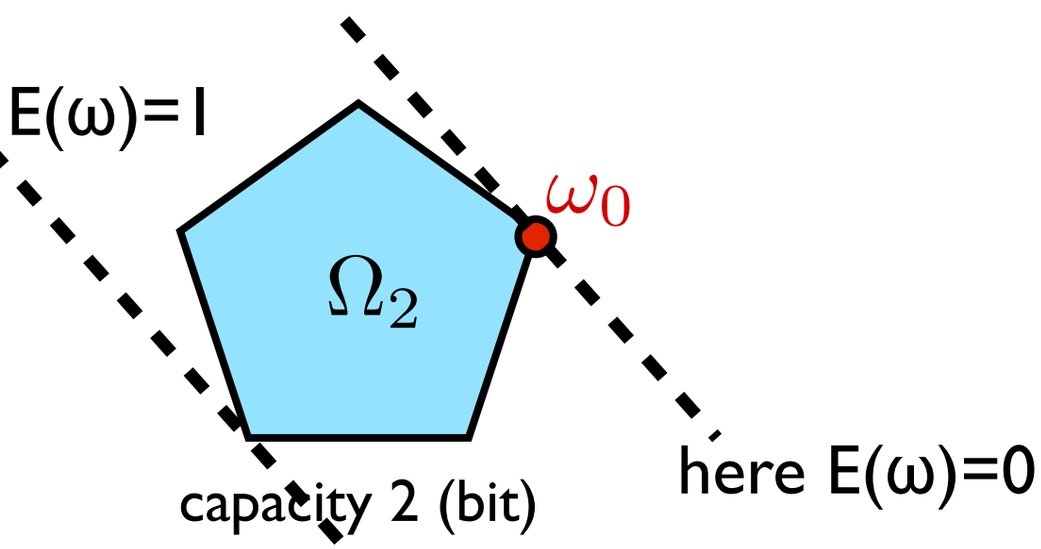
If there is a face, similar reasoning:

Ω_1 contains ∞ many states. ✗



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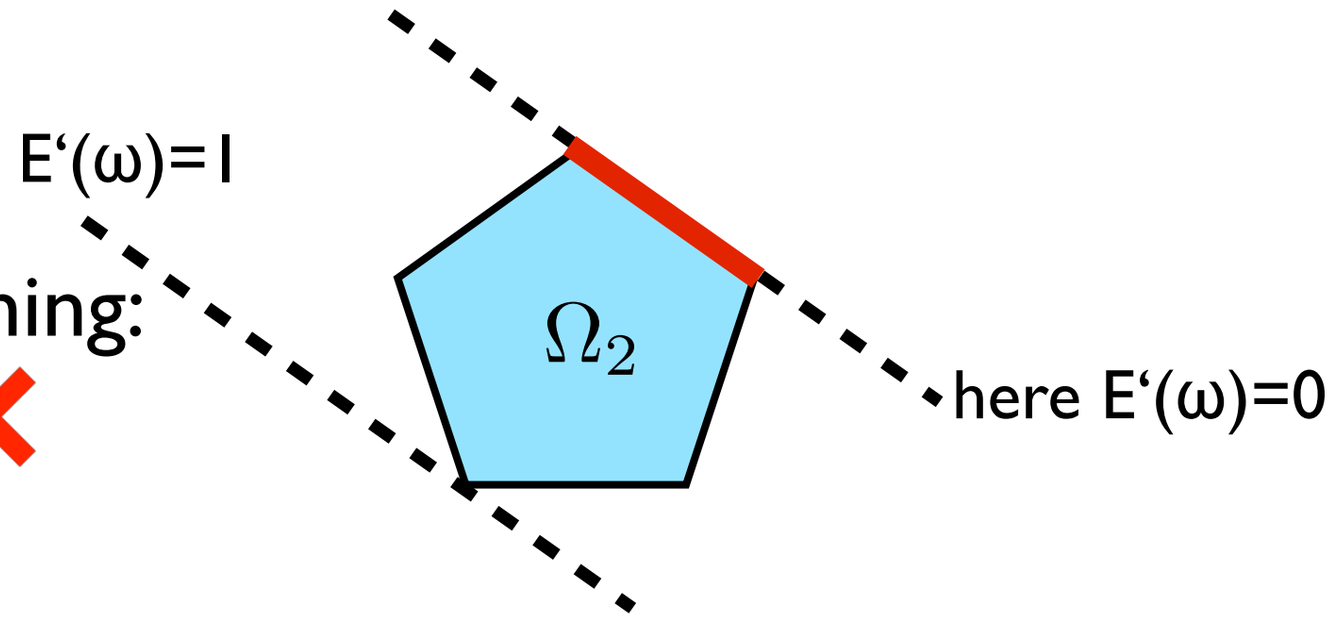
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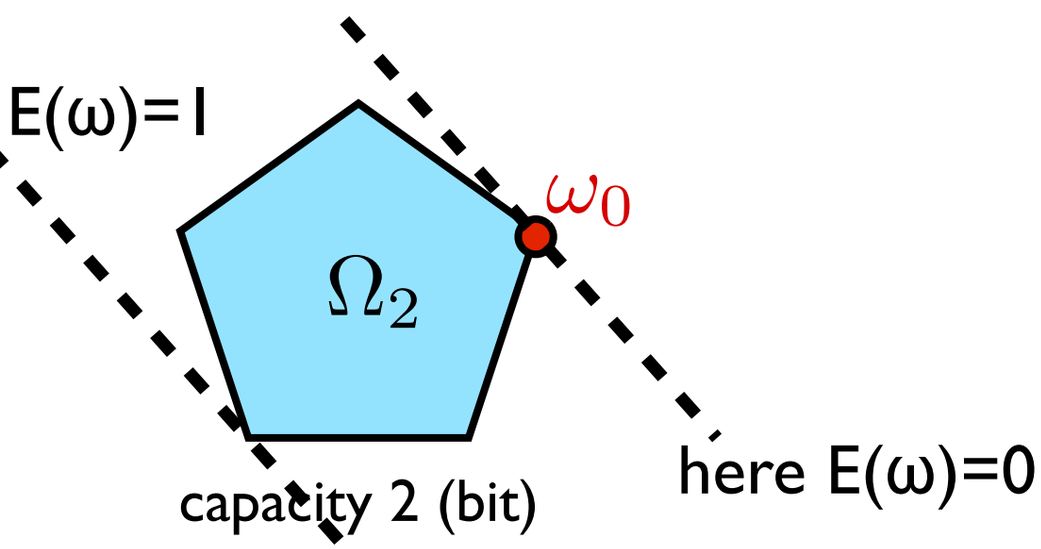


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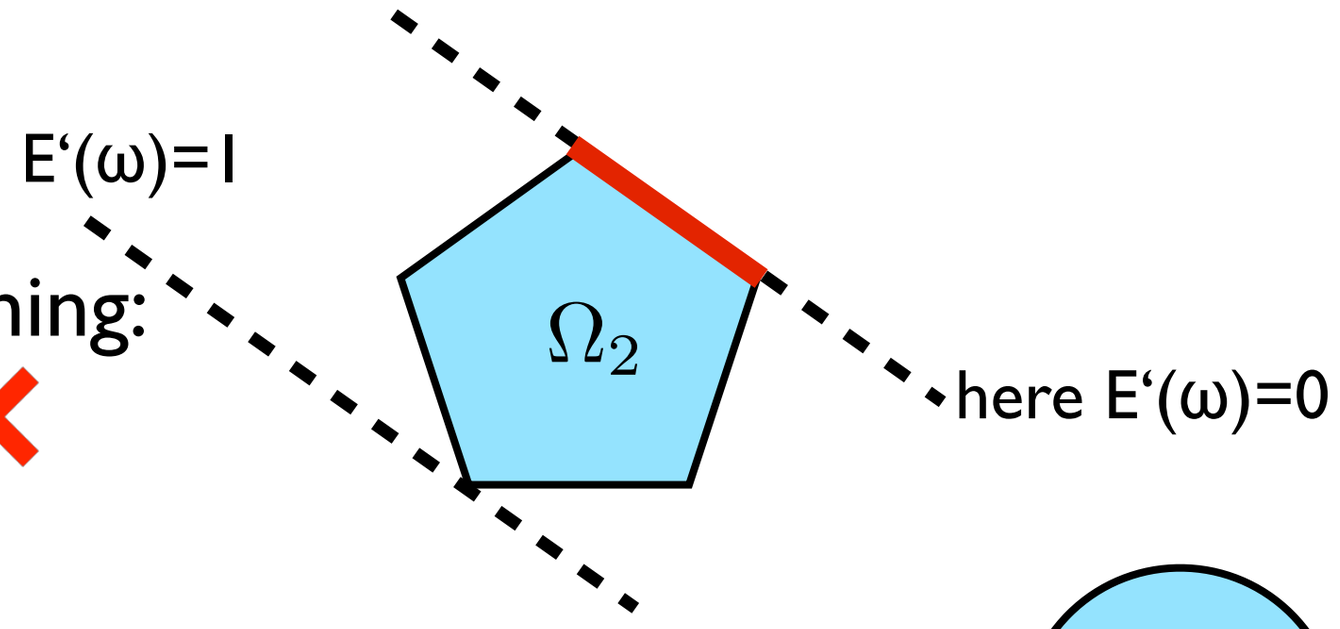
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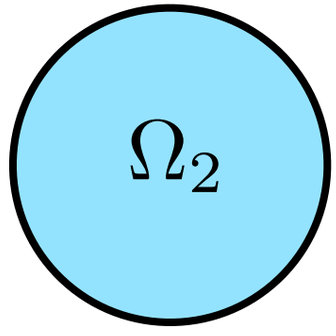
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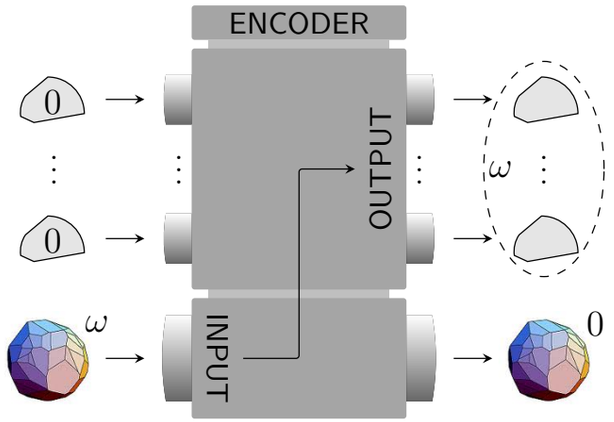
⇒no faces: 

Reversibility axiom ⇒ Ω_2 is a ball. 

Overview

1. An (incomplete and physicists') historical perspective

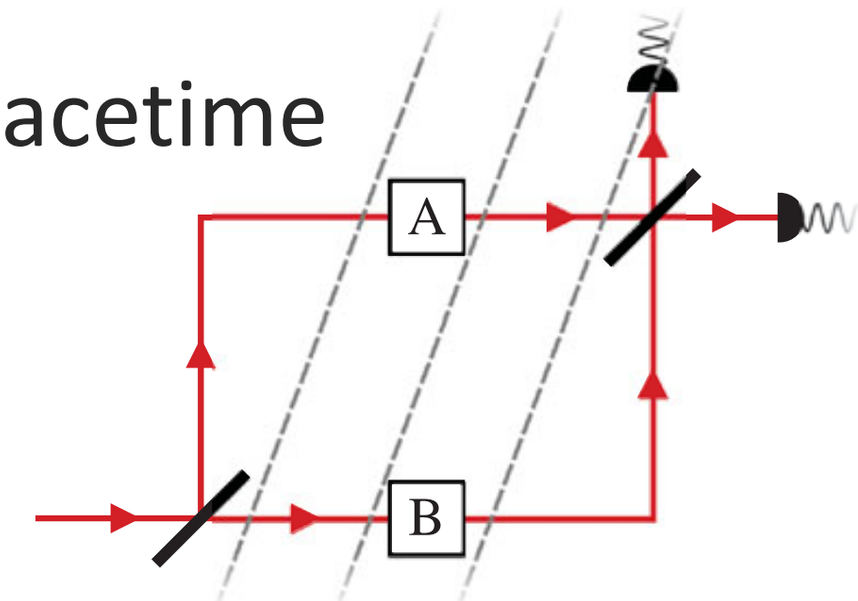
2. Generalized probabilistic theories



3. An example reconstruction of QT

4. Beyond reconstructions: QT and spacetime

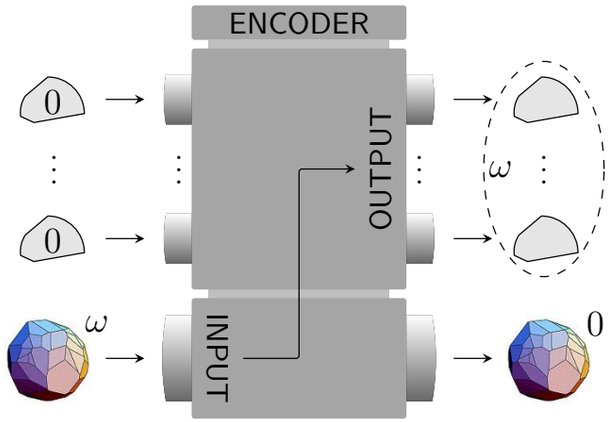
5. Conclusion



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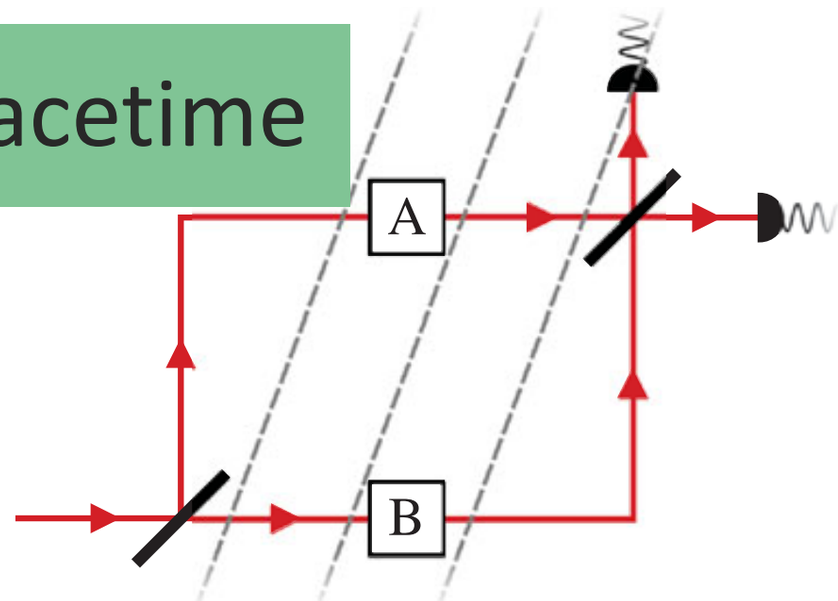
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5. Conclusion



The qubit revisited

We have seen: simple assumptions tell us that a **bit** should have a **Euclidean ball** state space.



$d = 1$

classical bit

$d = 2$



$d = 3$

quantum bit

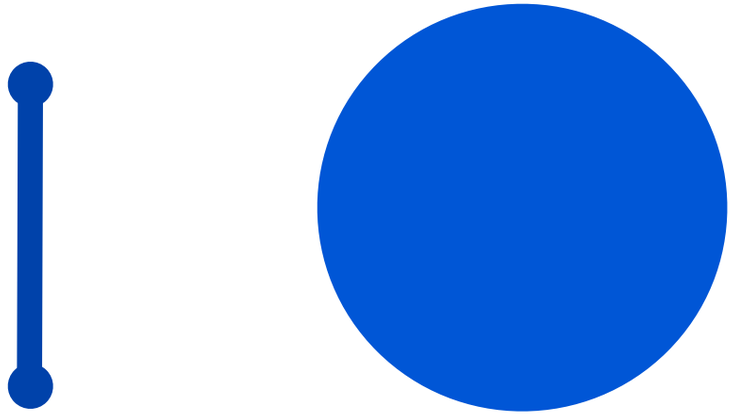


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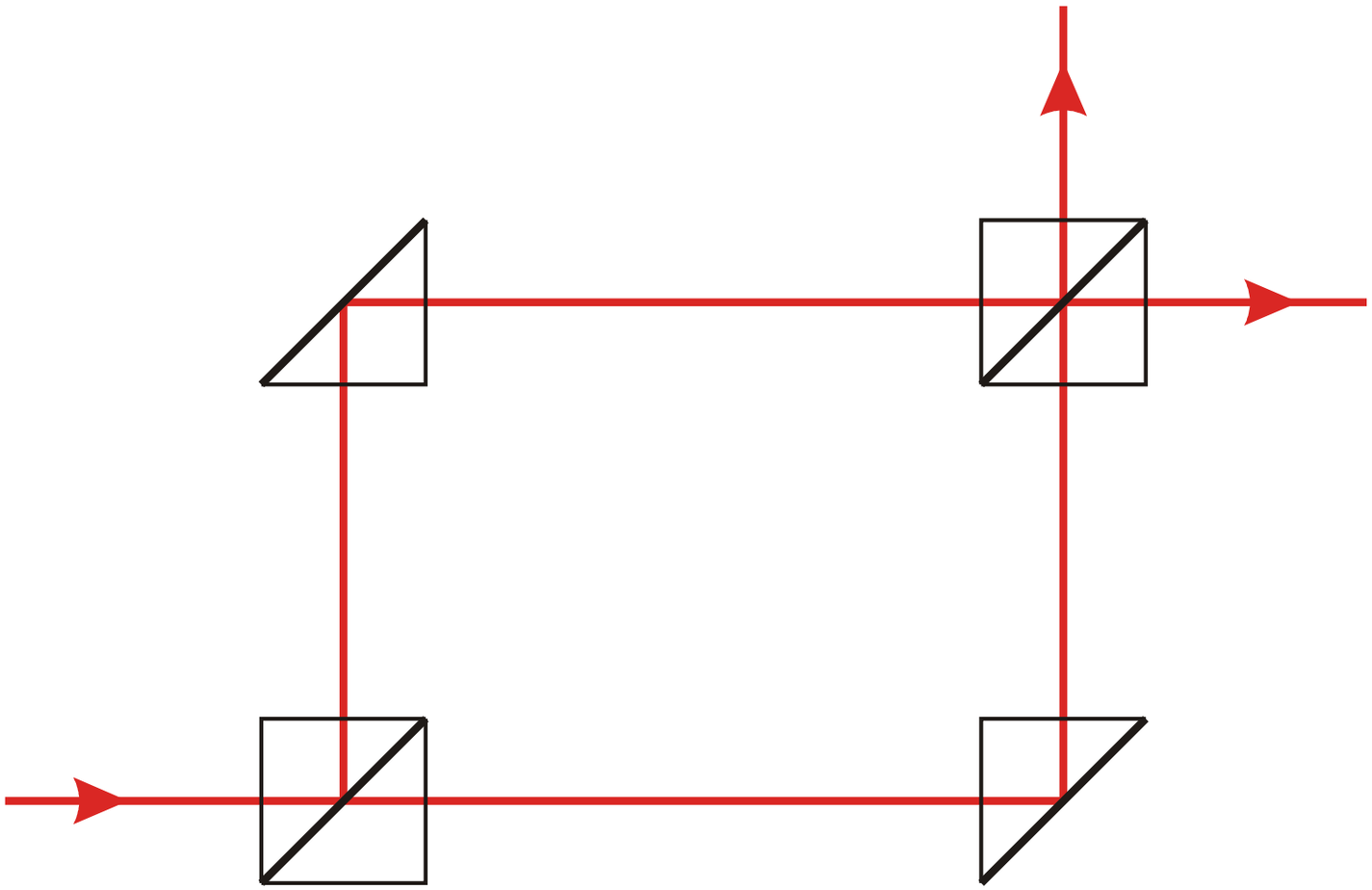
$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ -qubits would have $d = 2, 3, 5, 9.$

Why $d = 3$?

We have already seen an **information-theoretic** reason.
But there is also a “spacetime” reason!

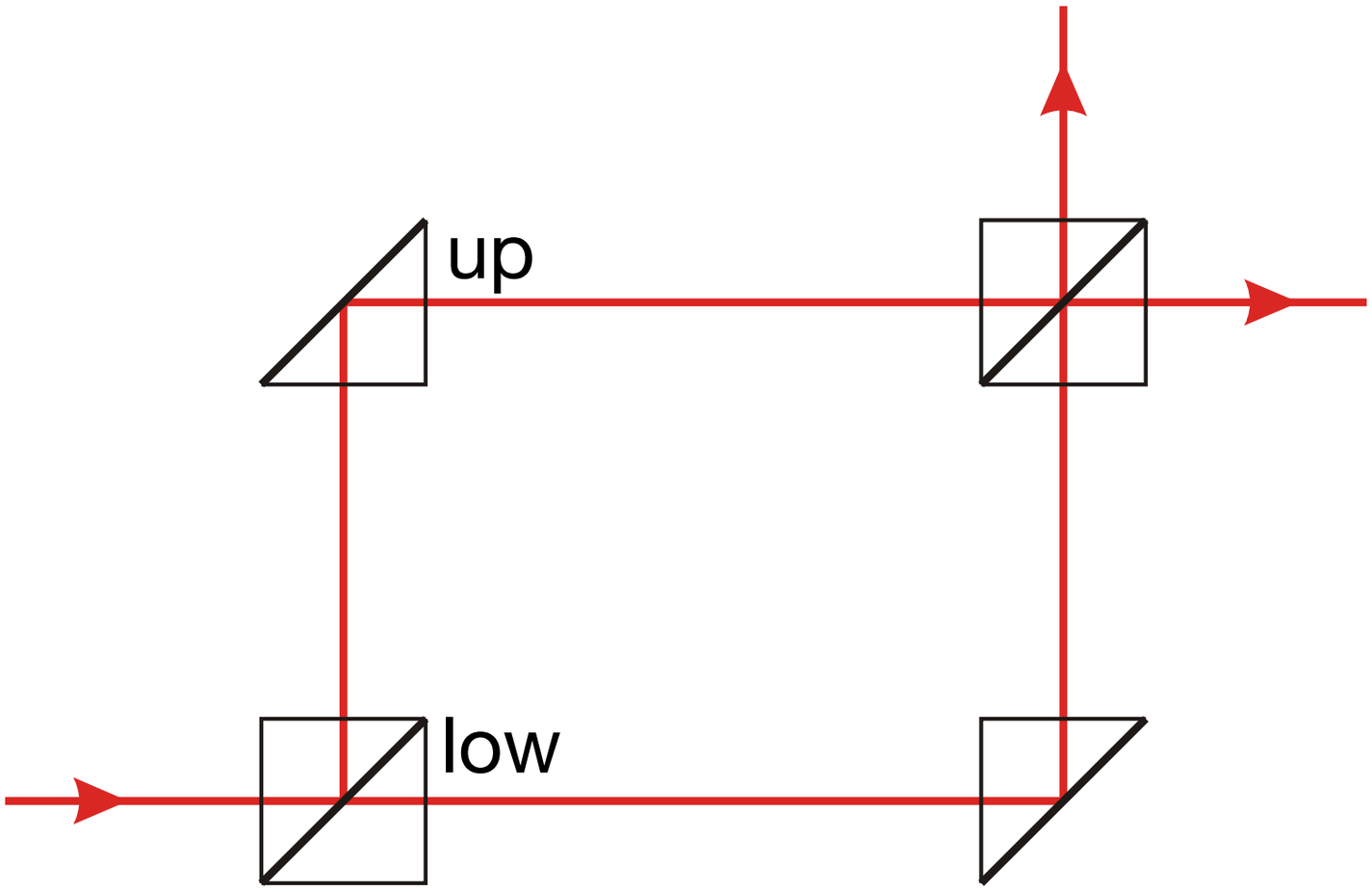
Constraints from relativity

A. Garner, **MM**, O. C. O. Dahlsten, Proc. R. Soc. A **473**, 20170596 (2017).



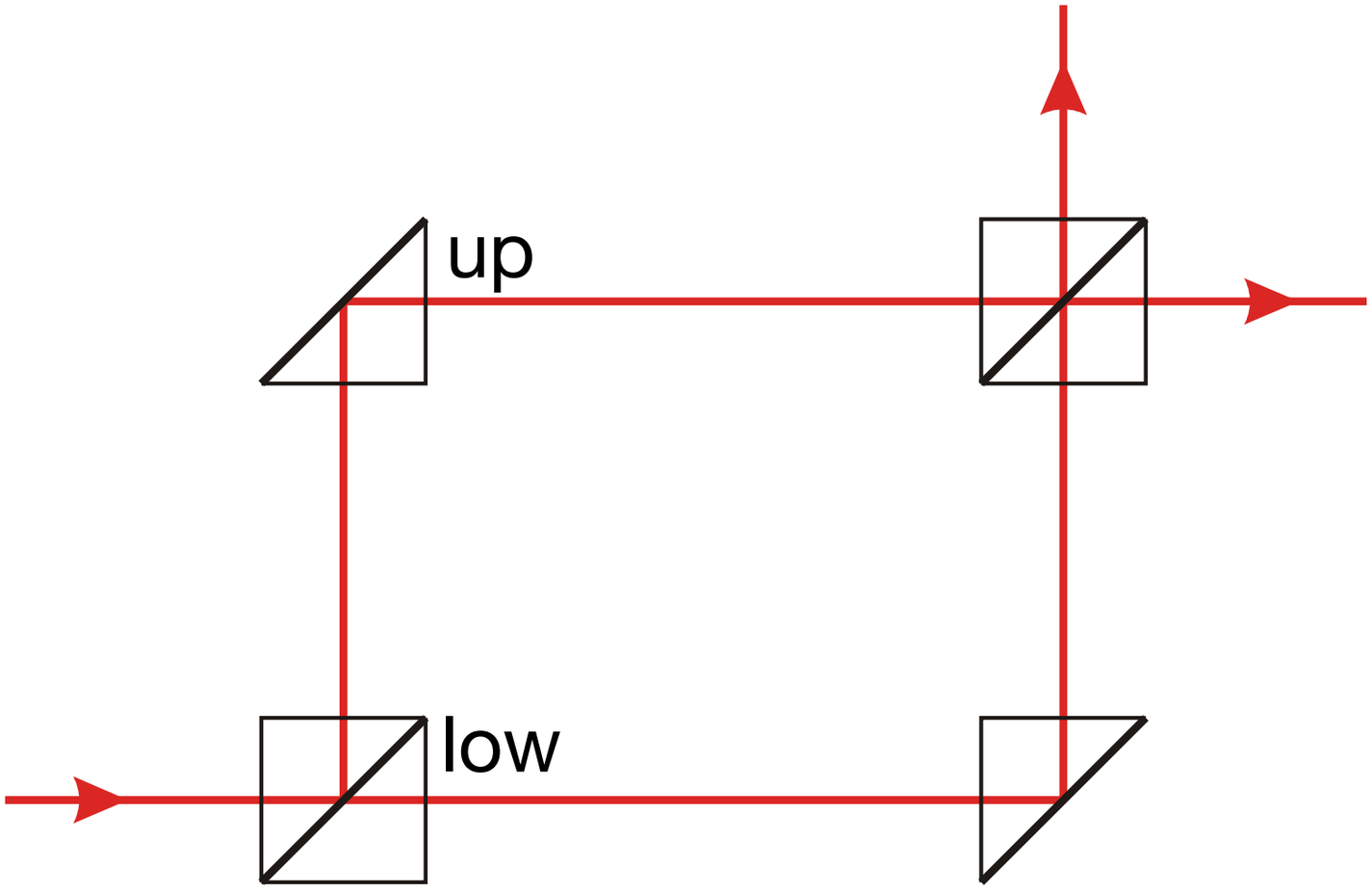
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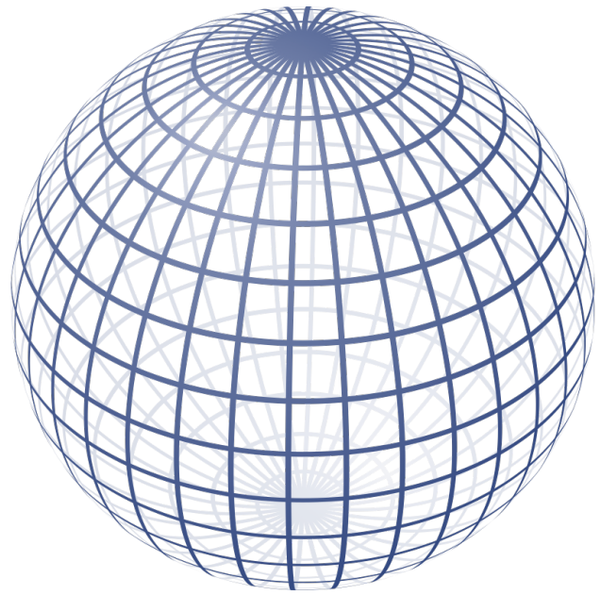
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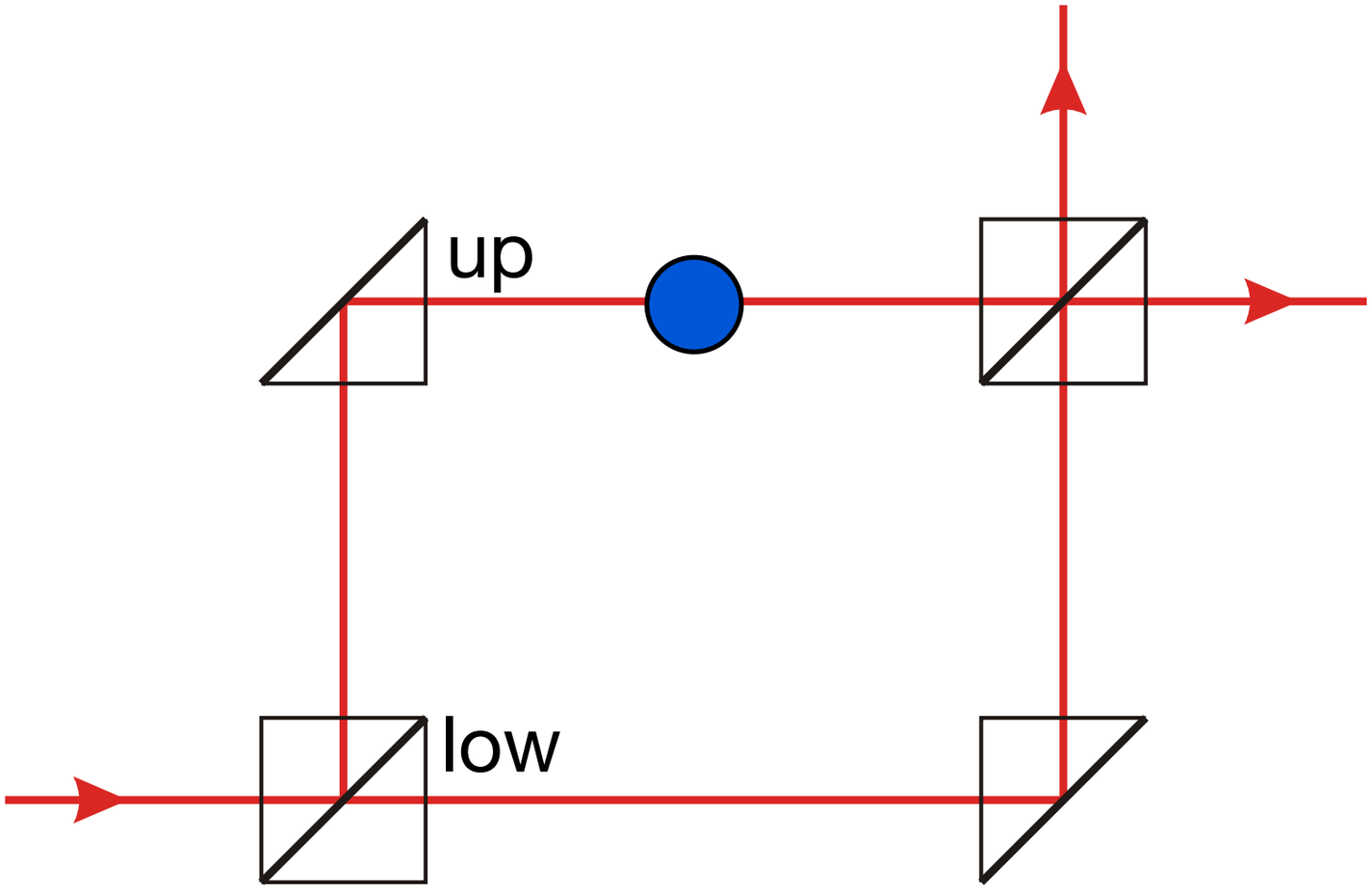


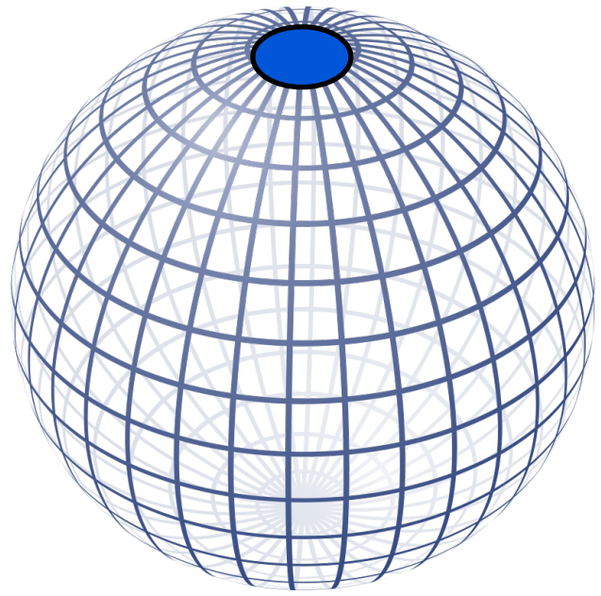


d-dim. "Bloch sphere"

Constraints from relativity

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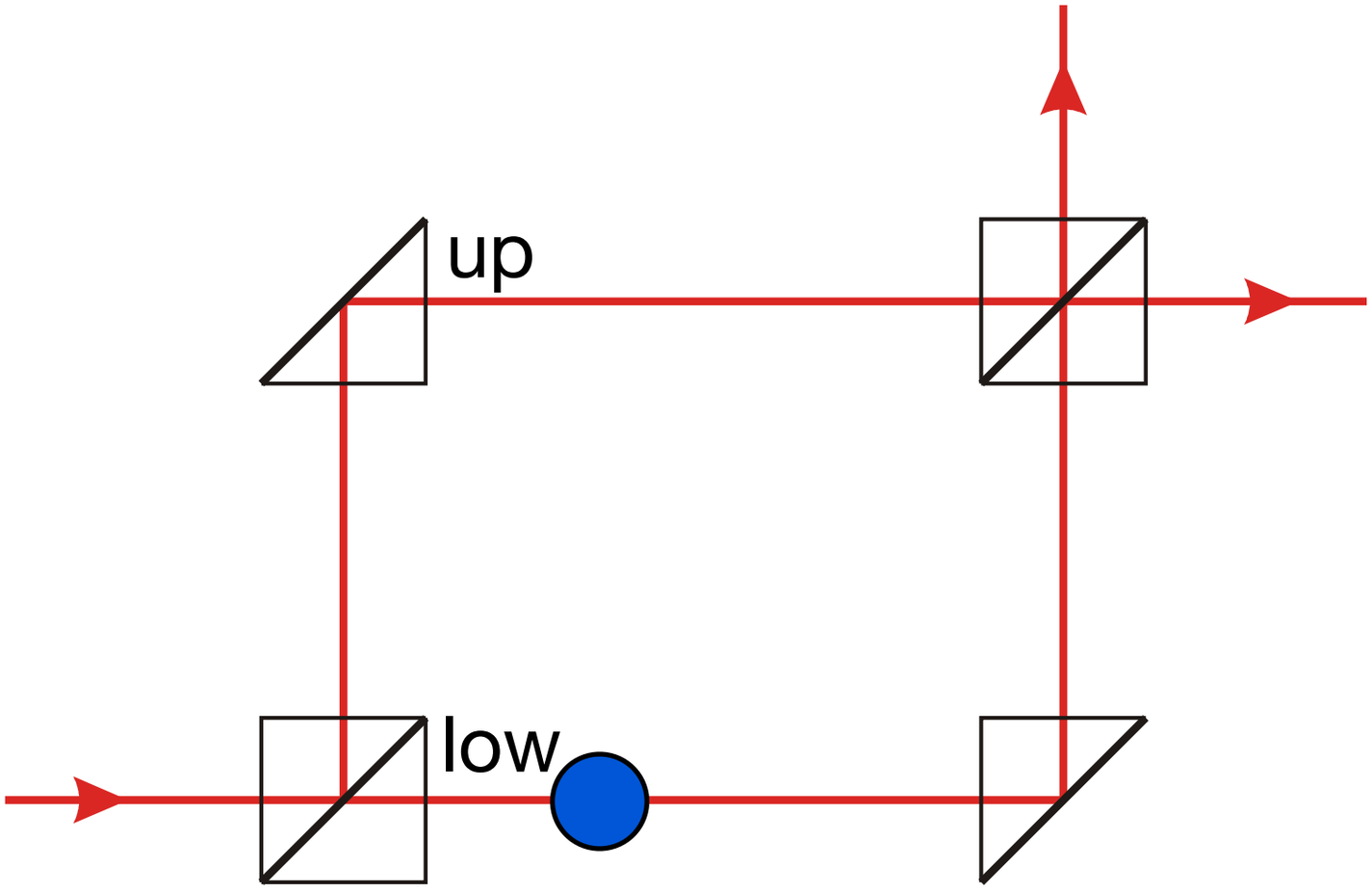


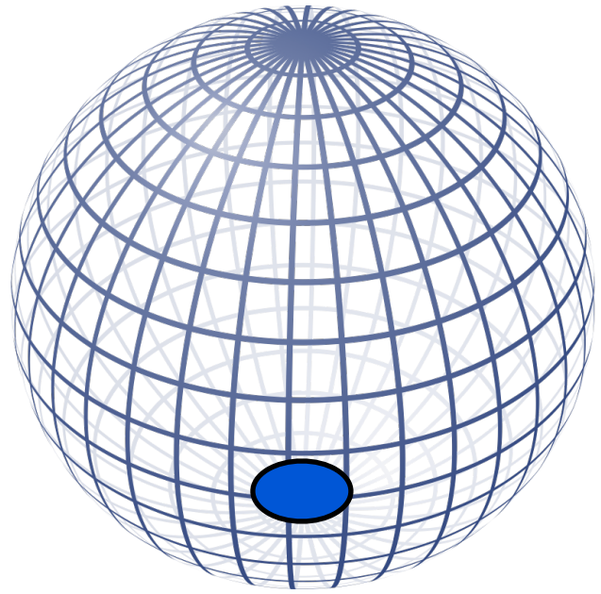
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North-pole state: particle definitely in upper branch.

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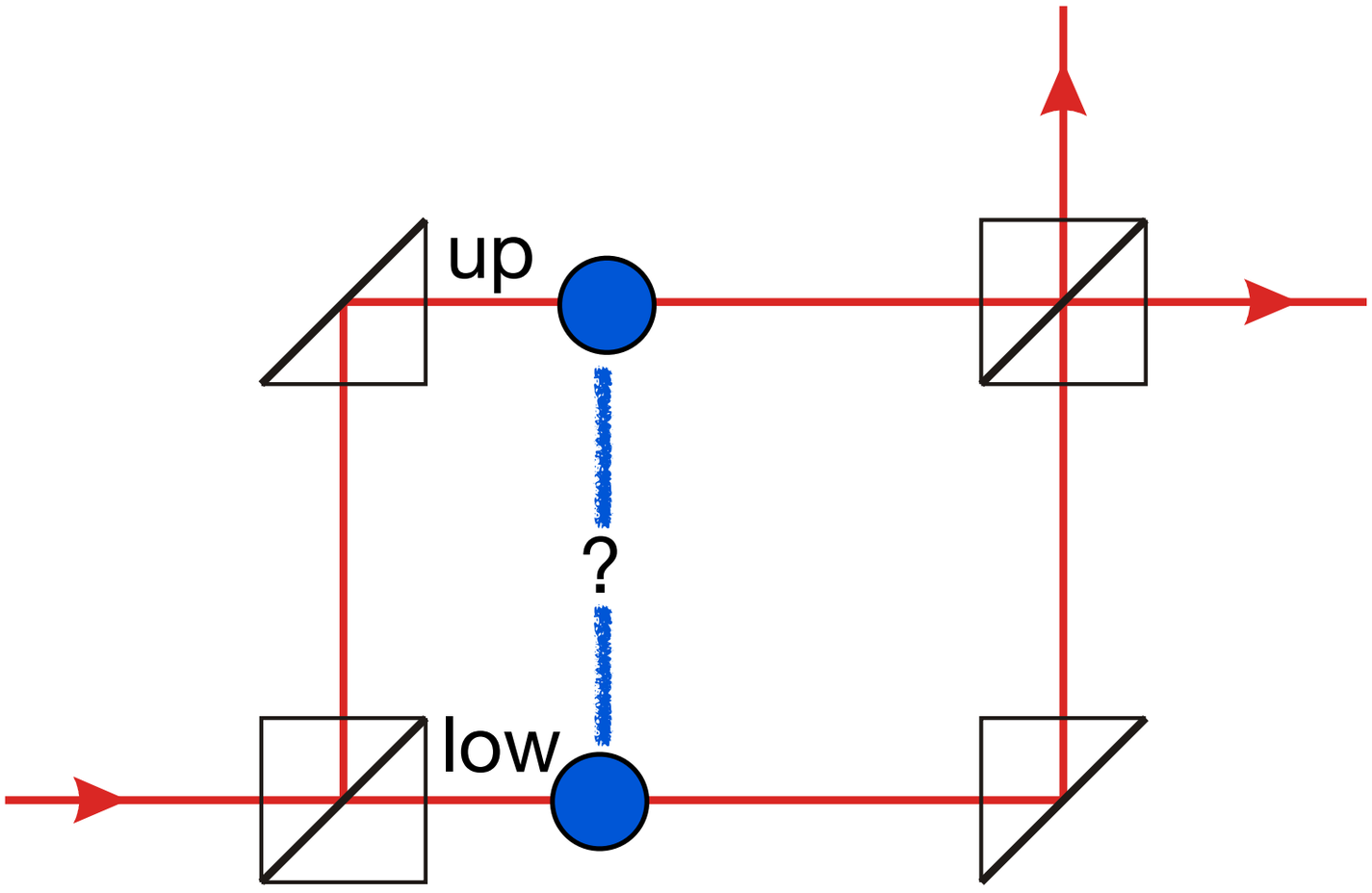


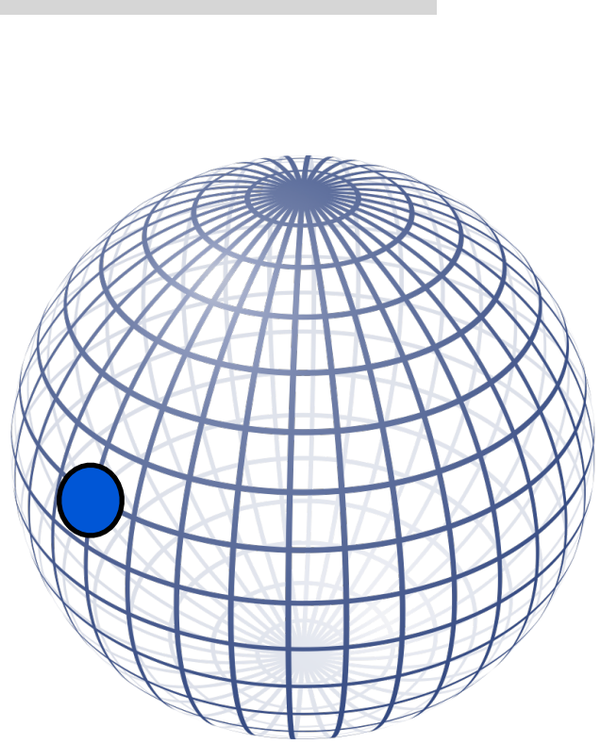
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South-pole state: particle definitely in lower branch.

Constraints from relativity

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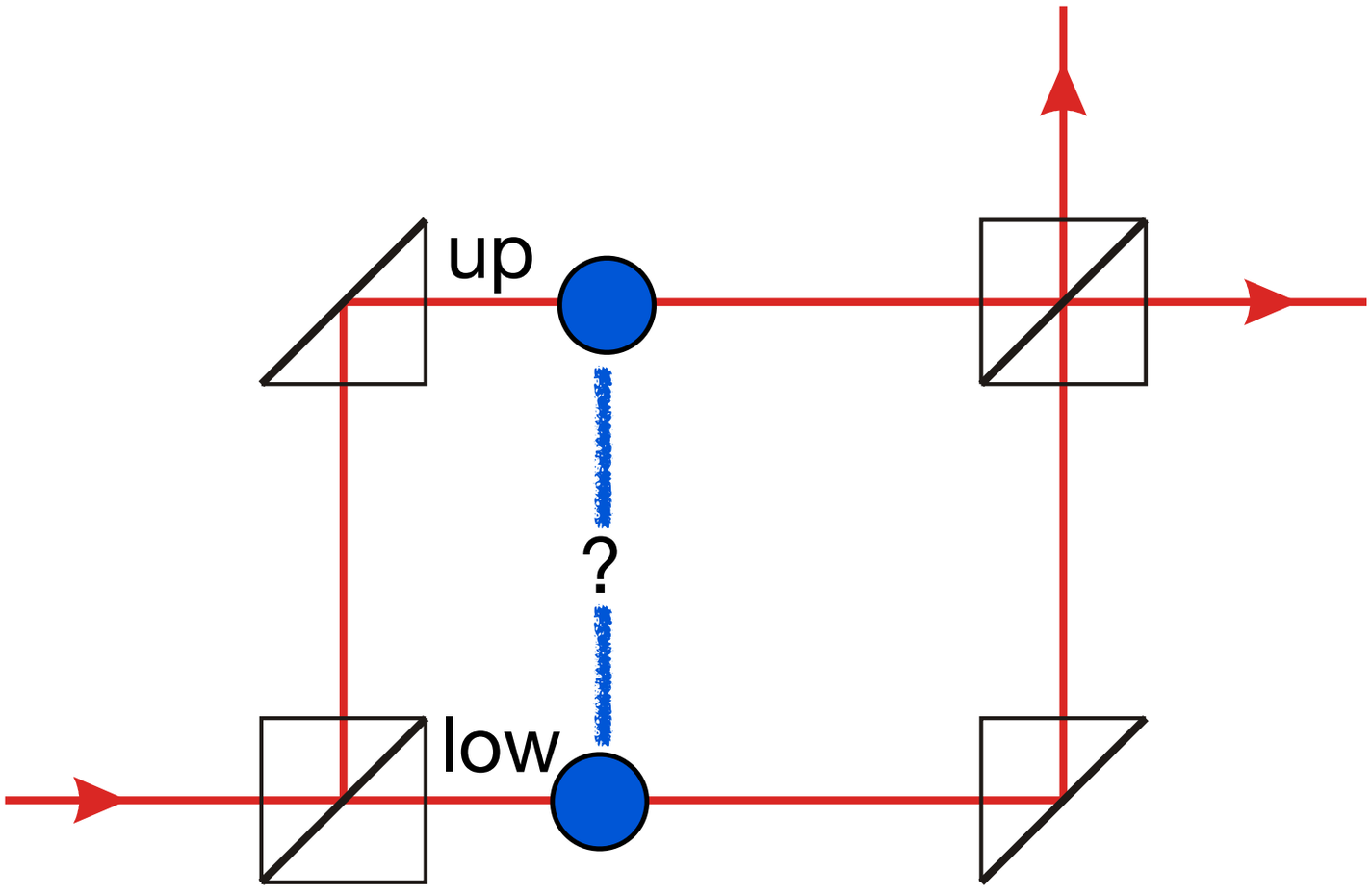


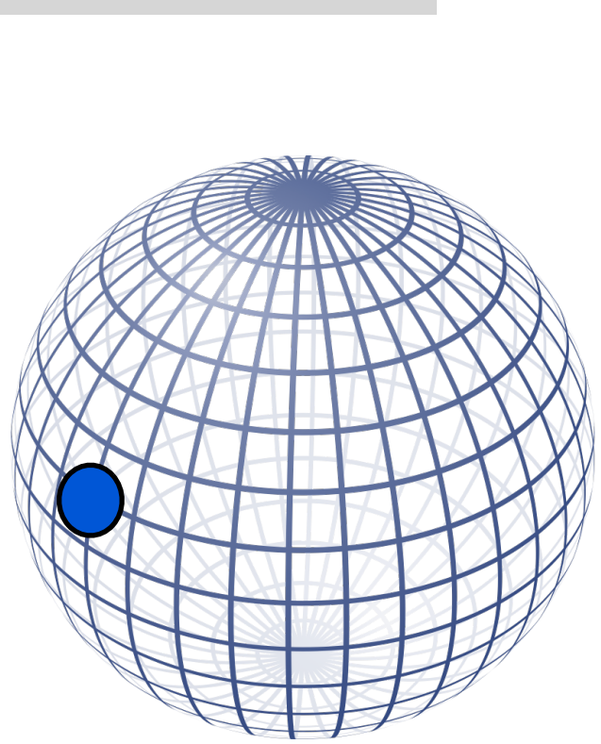
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State on equator *z=0*: probability 1/2 for each.

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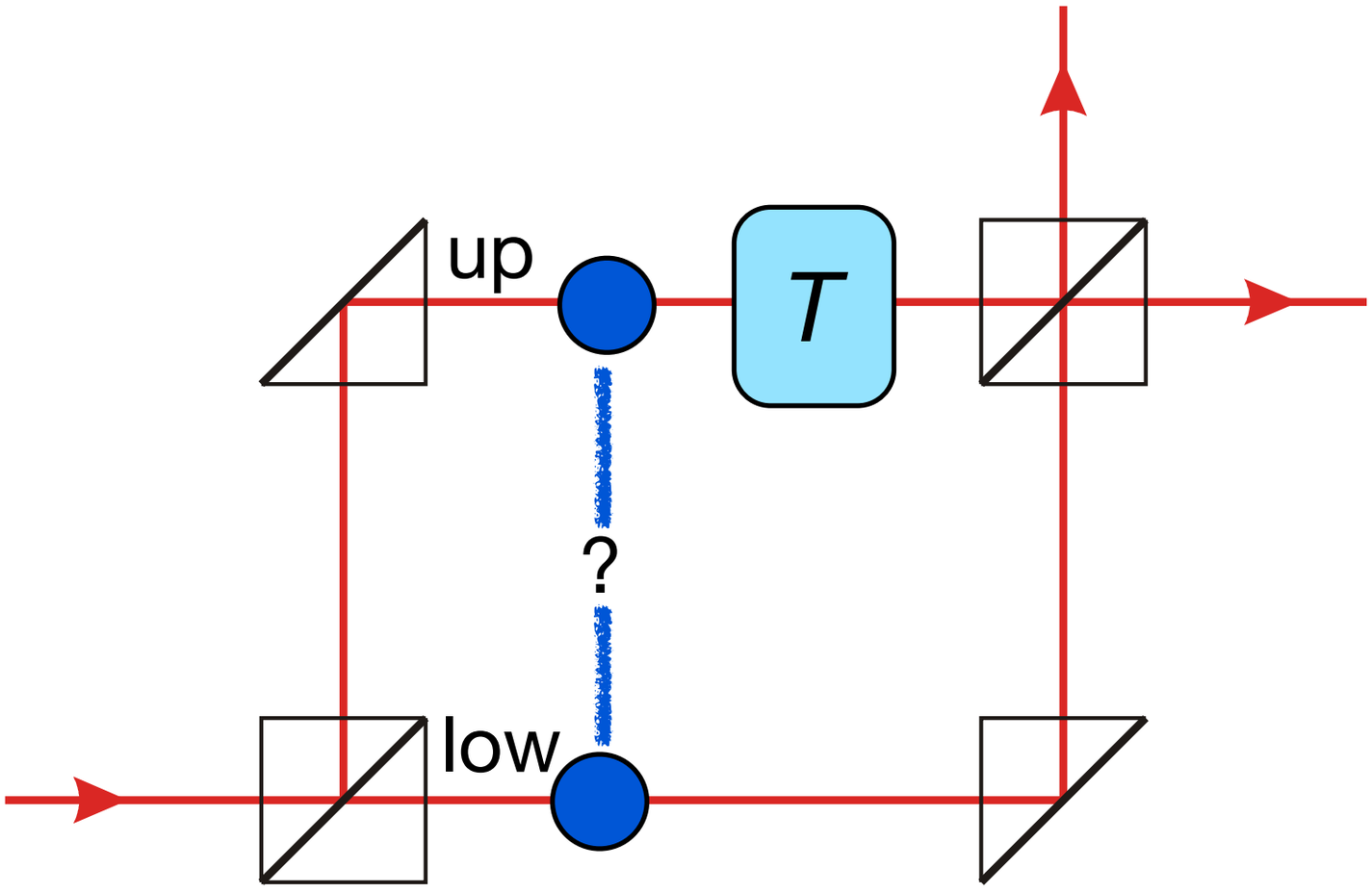
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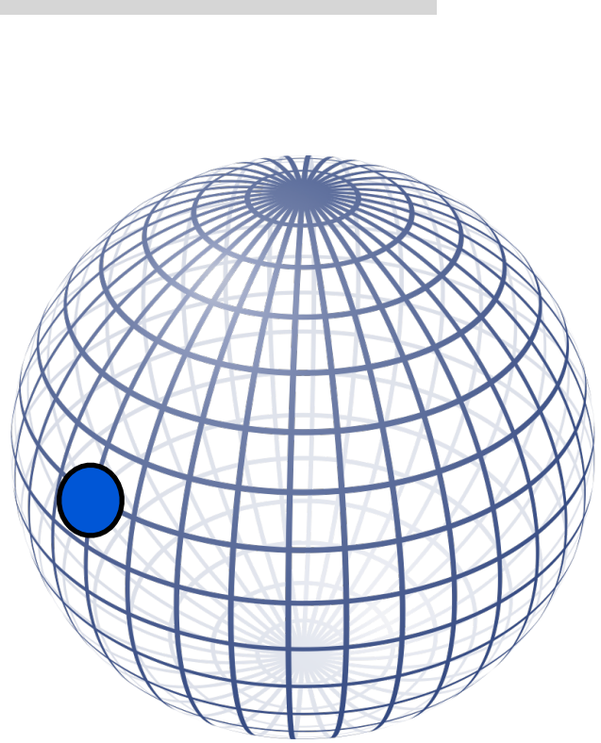
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$$p(\mathrm{up}) = \tfrac{1}{2}(z+1)$$

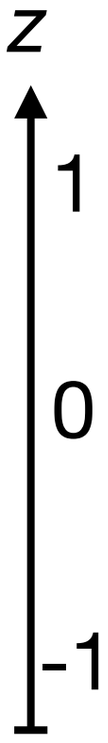
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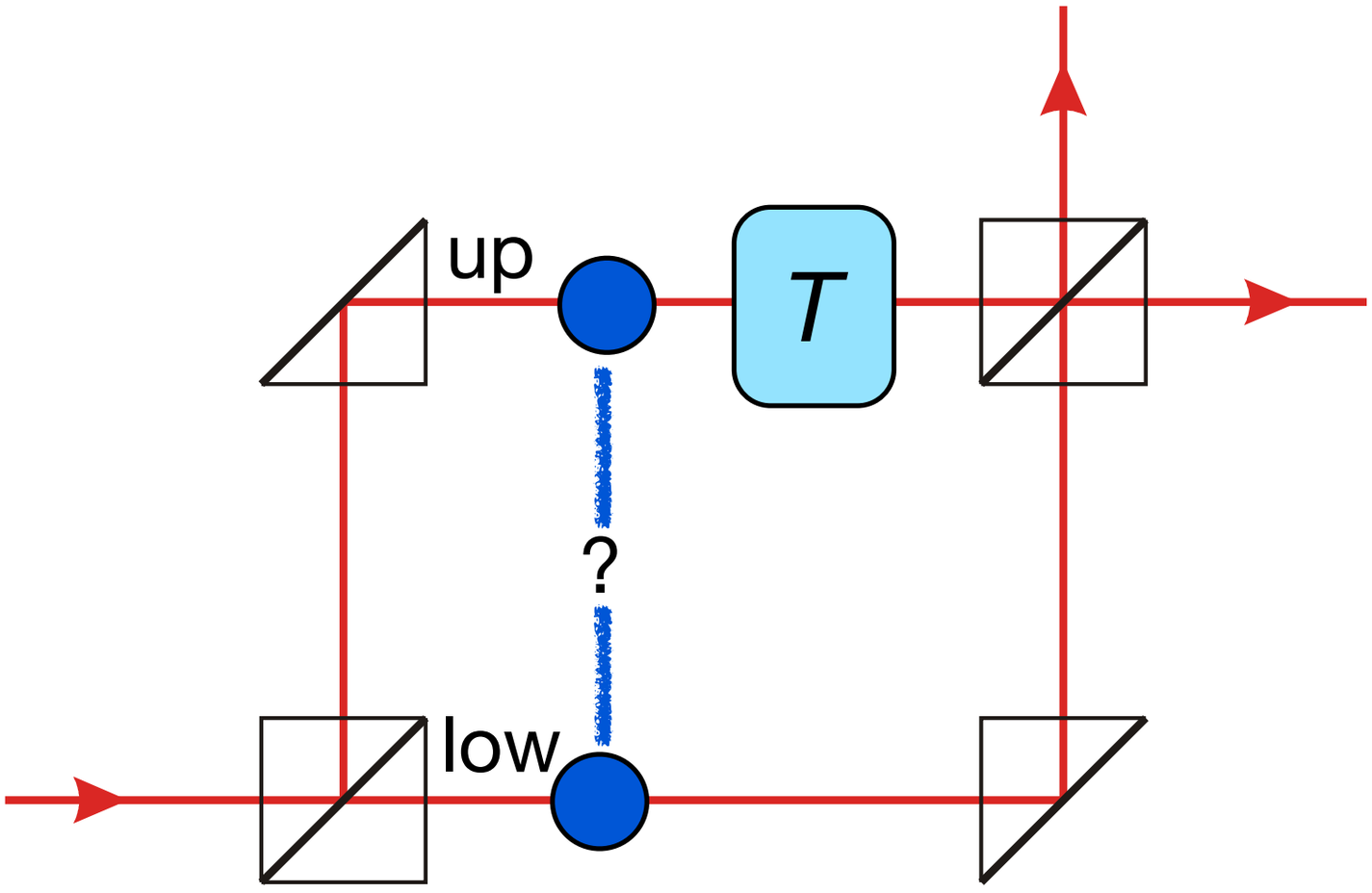


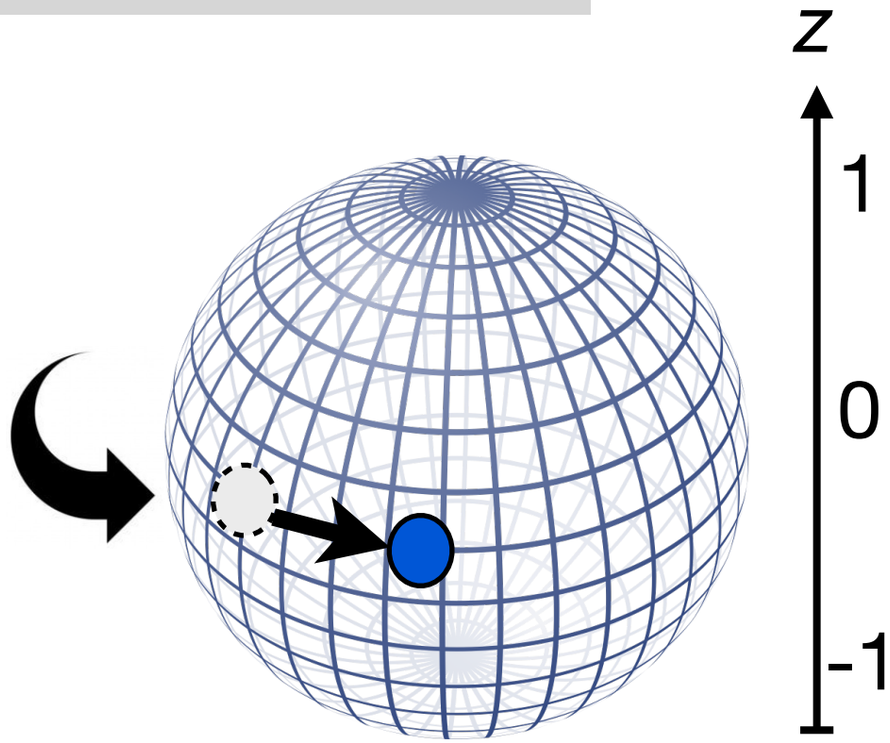
What transformations *T* can we perform locally in one arm...

... reversibly, i.e. without any information loss?

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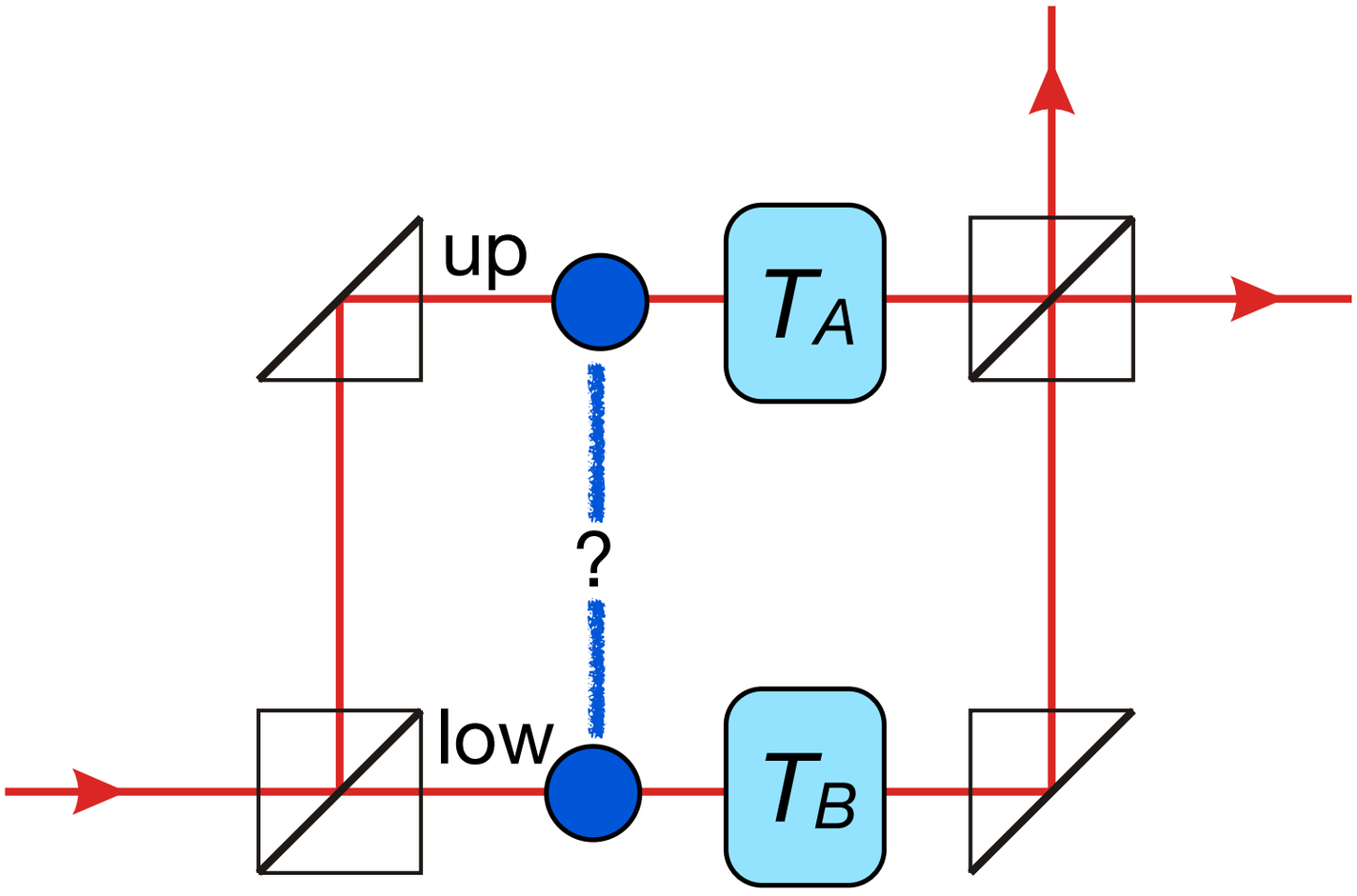
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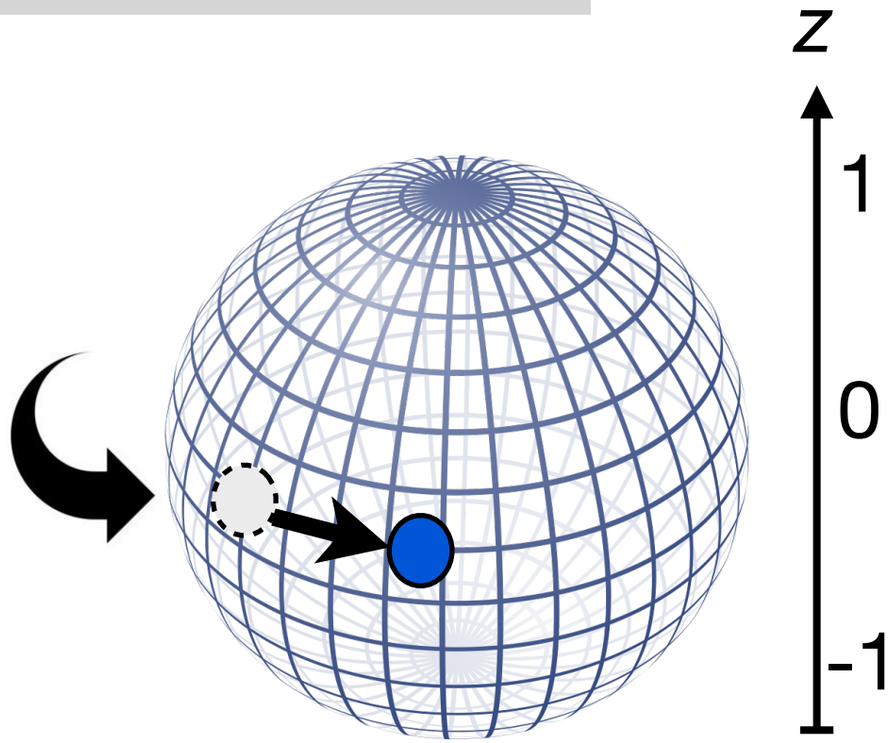
T must be a rotation of the Bloch ball (reversible+linear)...

... and must preserve *p*(up), i.e. preserve the *z*-axis.

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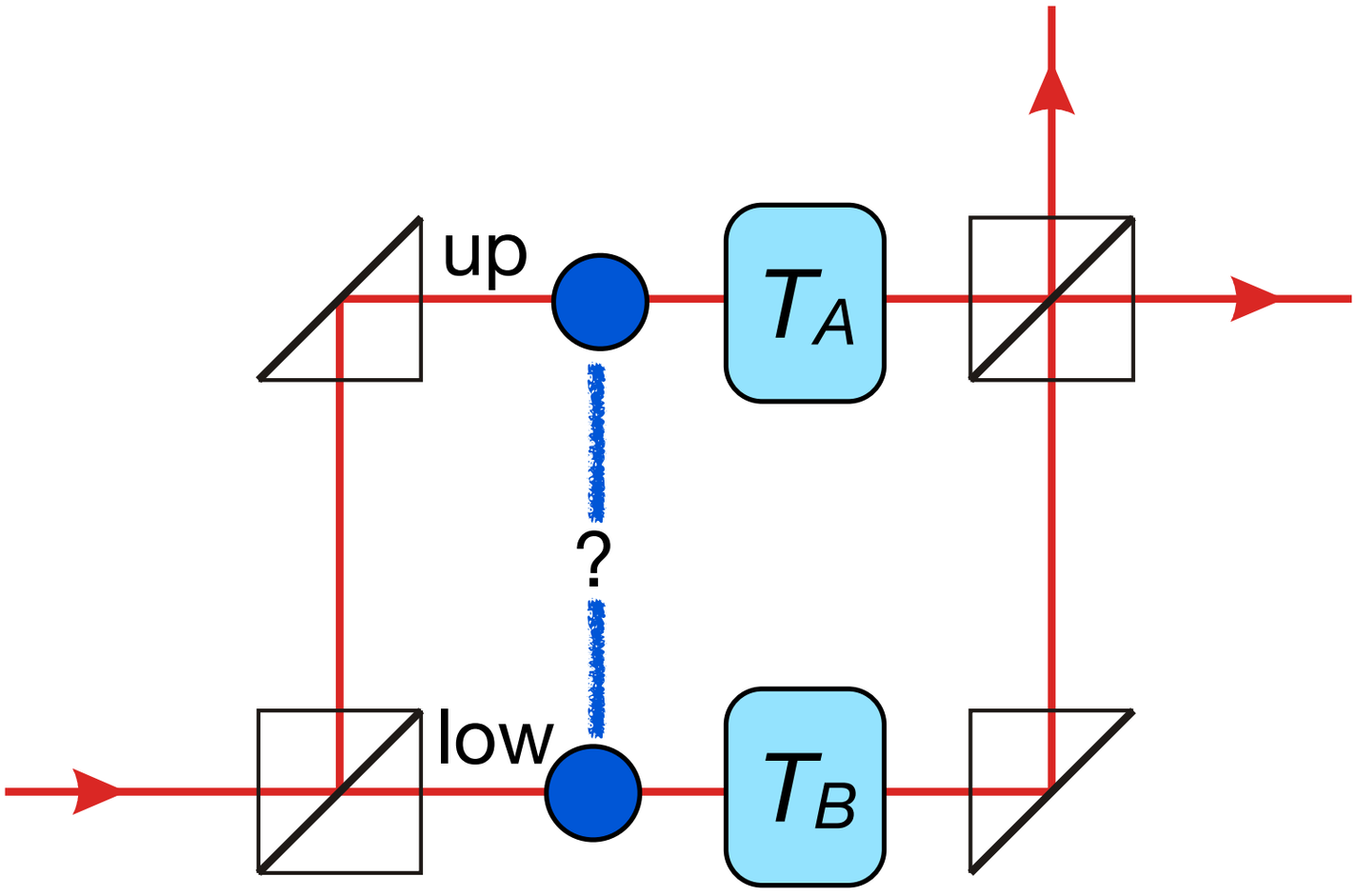
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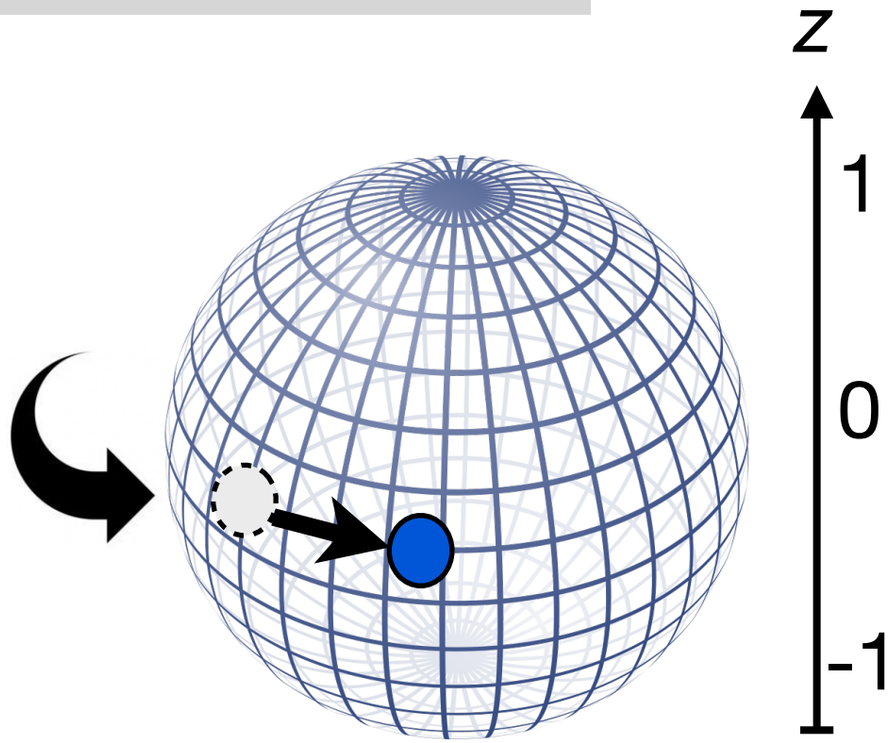
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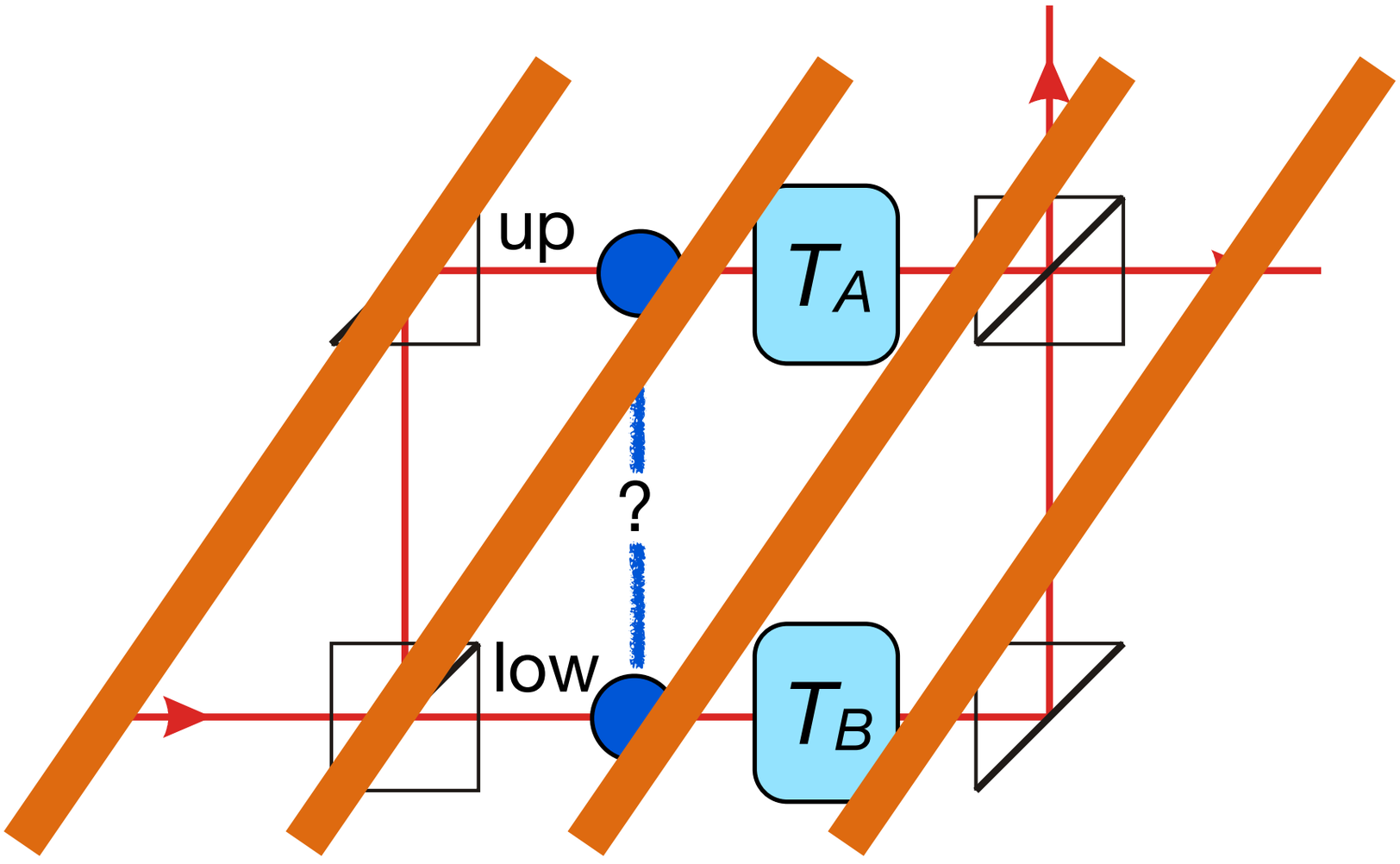
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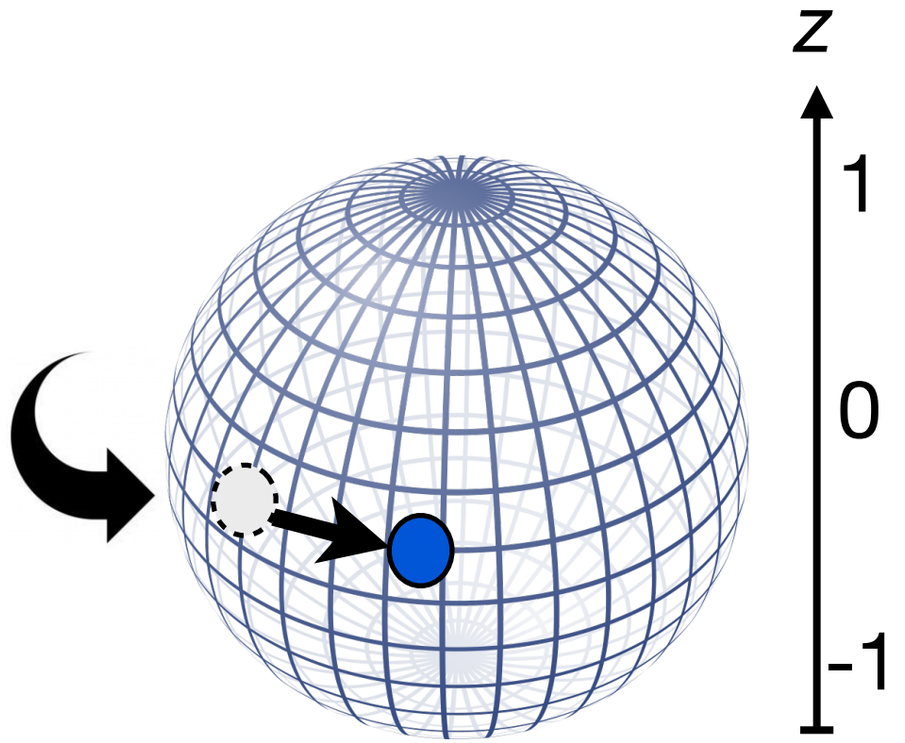
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$$\mathcal{G}_A = \mathcal{G}_B \simeq \mathrm{SO}(d-1).$$

Constraints from relativity



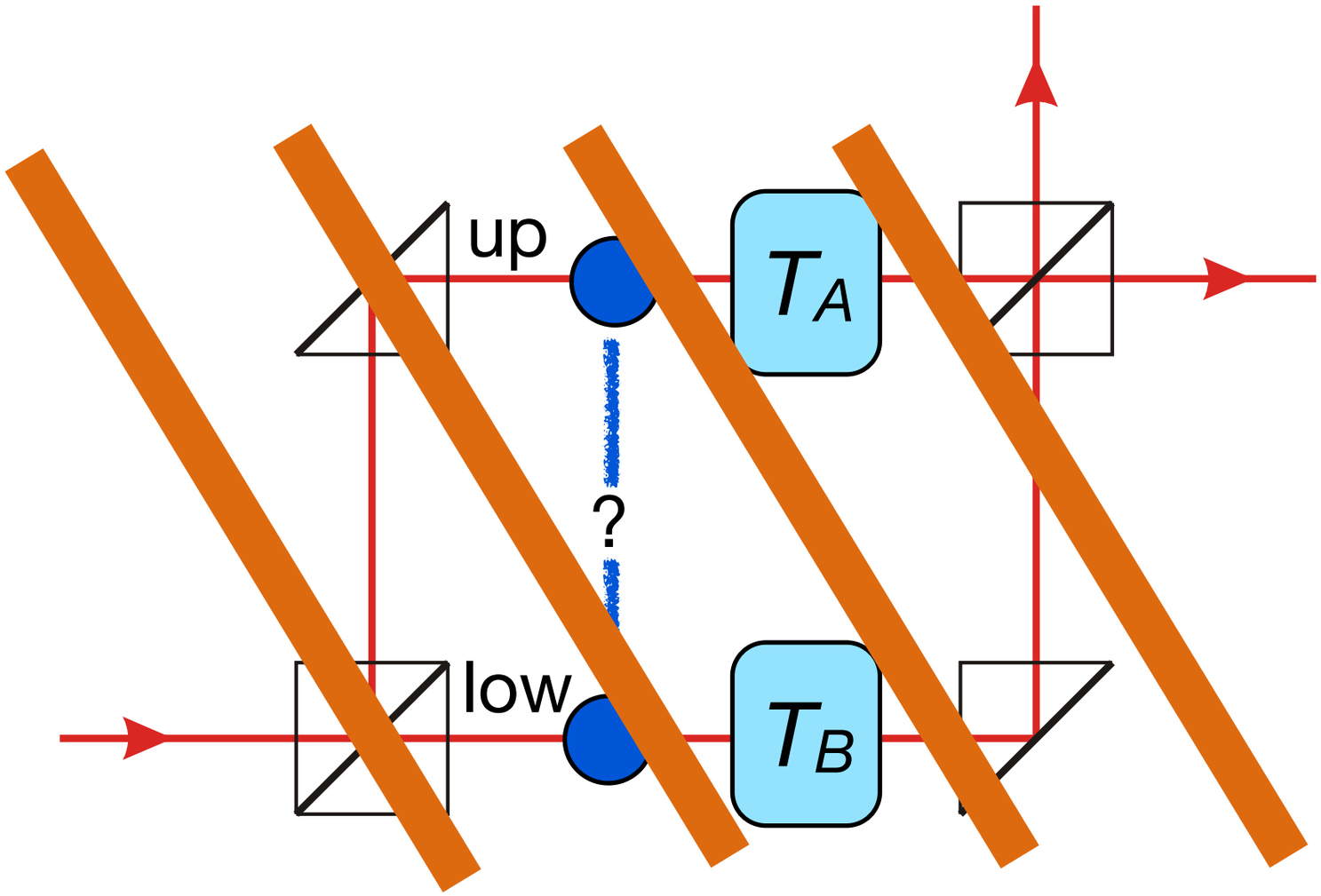


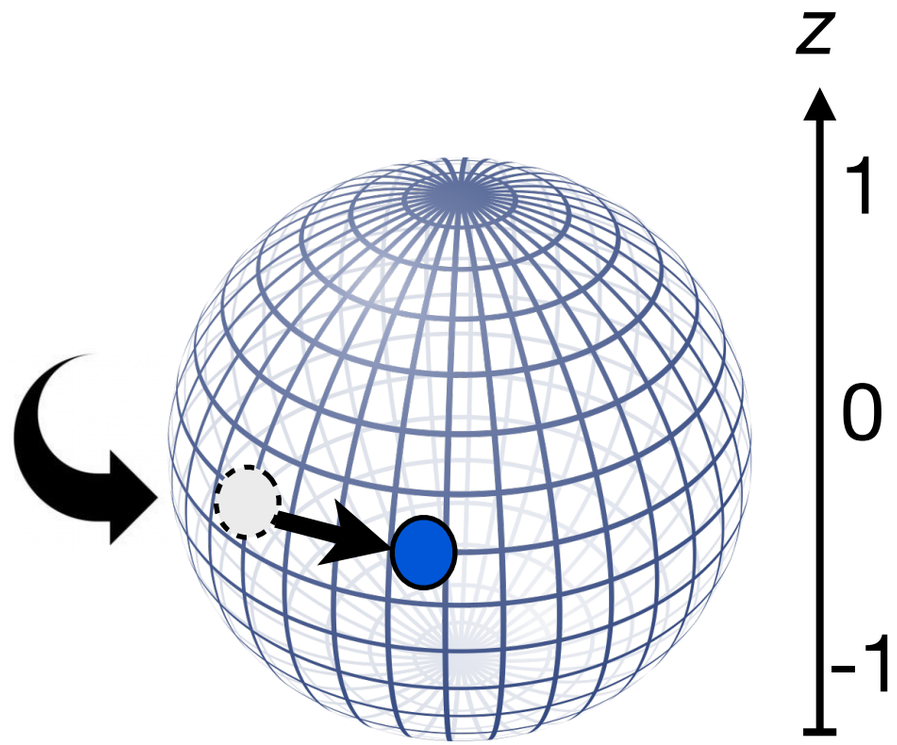
d-dim. "Bloch sphere"

Relativity: there's a frame of reference in which T_A happens before T_B...

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Constraints from relativity





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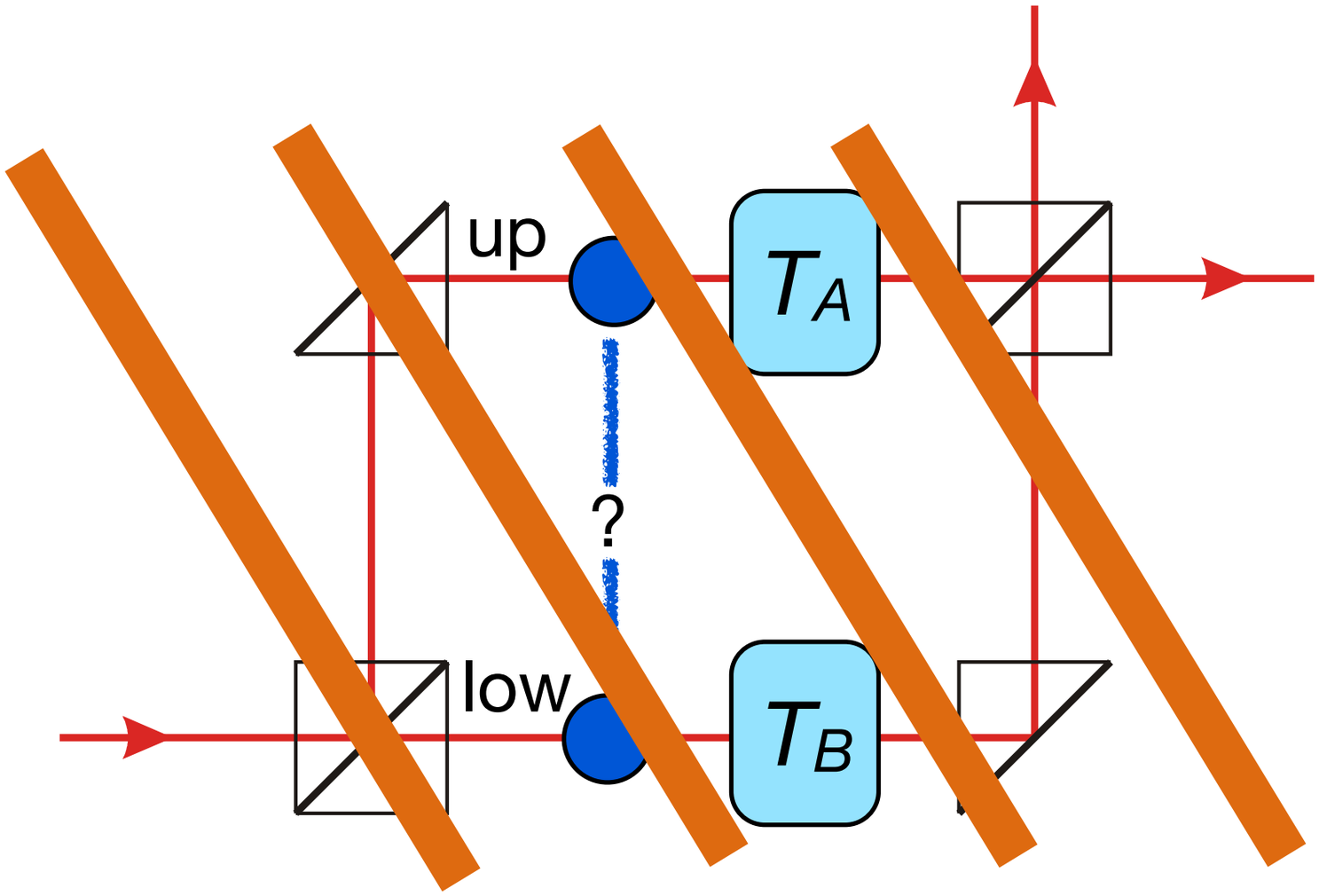
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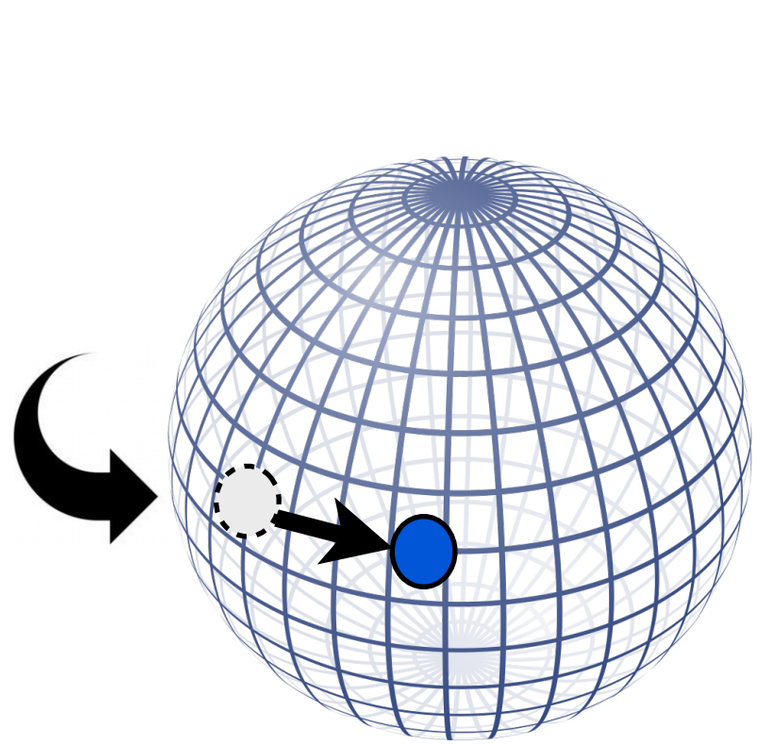
... and another frame where it's the other way around.

$$\mathcal{G}_A = \mathcal{G}_B \simeq \mathrm{SO}(d-1).$$

Constraints from relativity

$$\Rightarrow\ T_A T_B = T_B T_A \text{ for all } T_A, T_B \in \mathrm{SO}(d-1).$$





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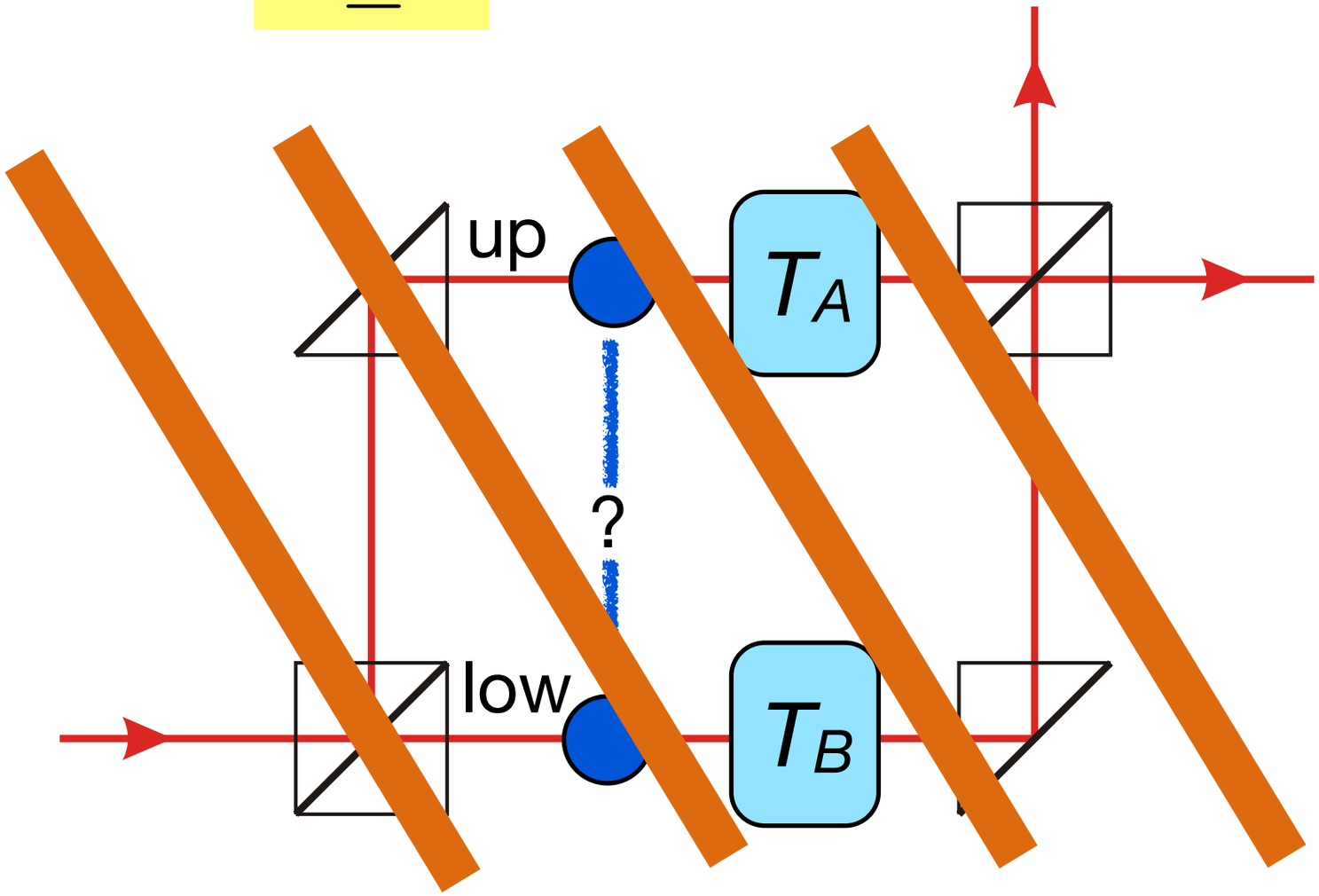
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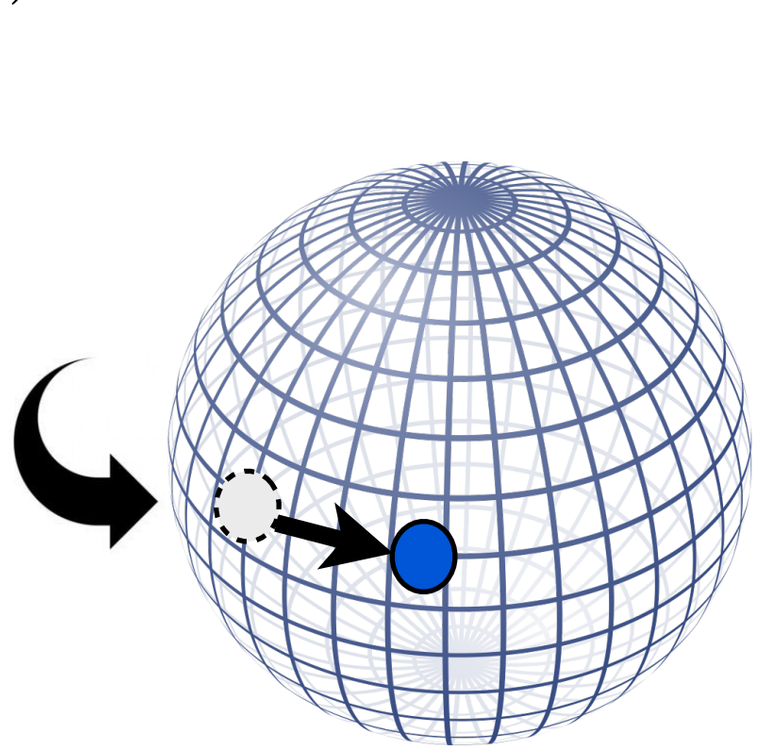
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Constraints from relativity

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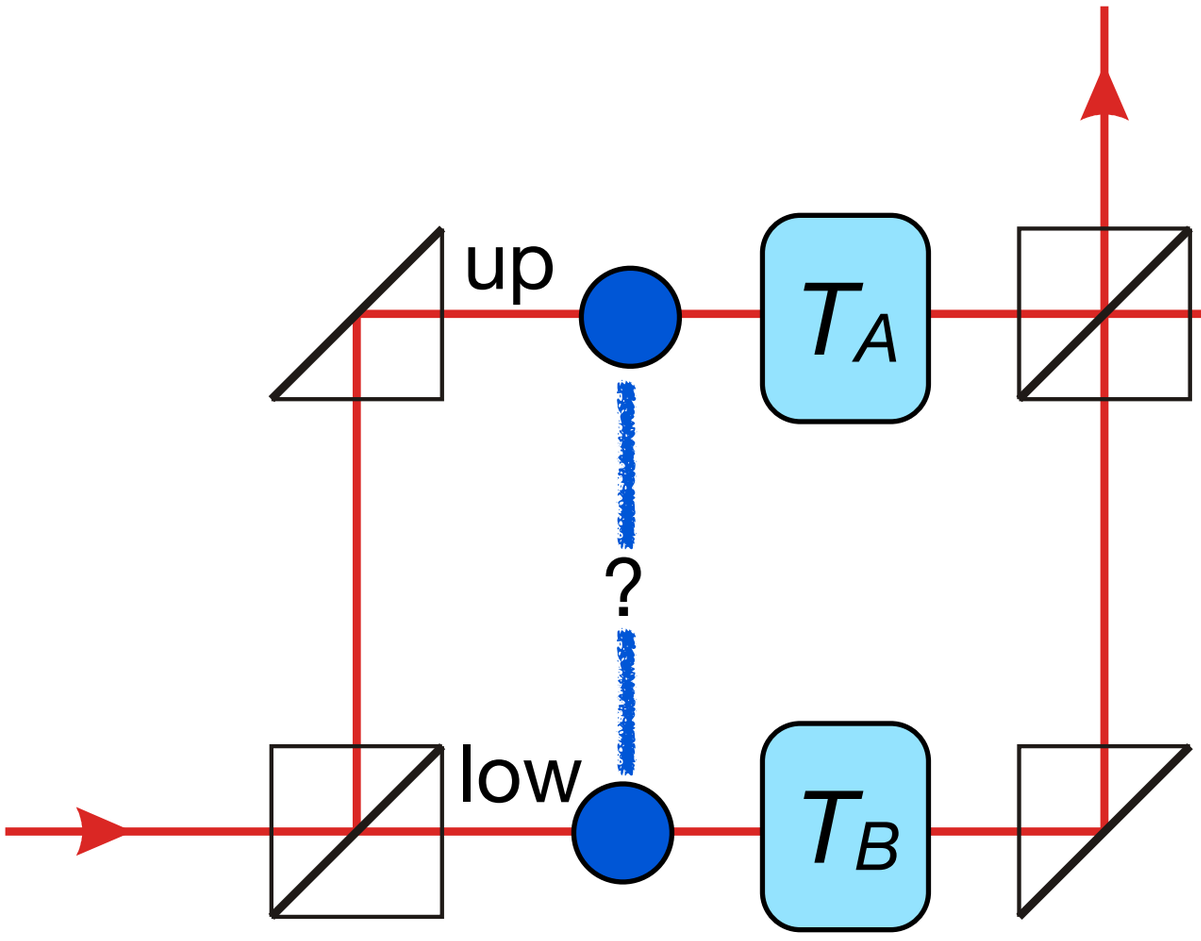
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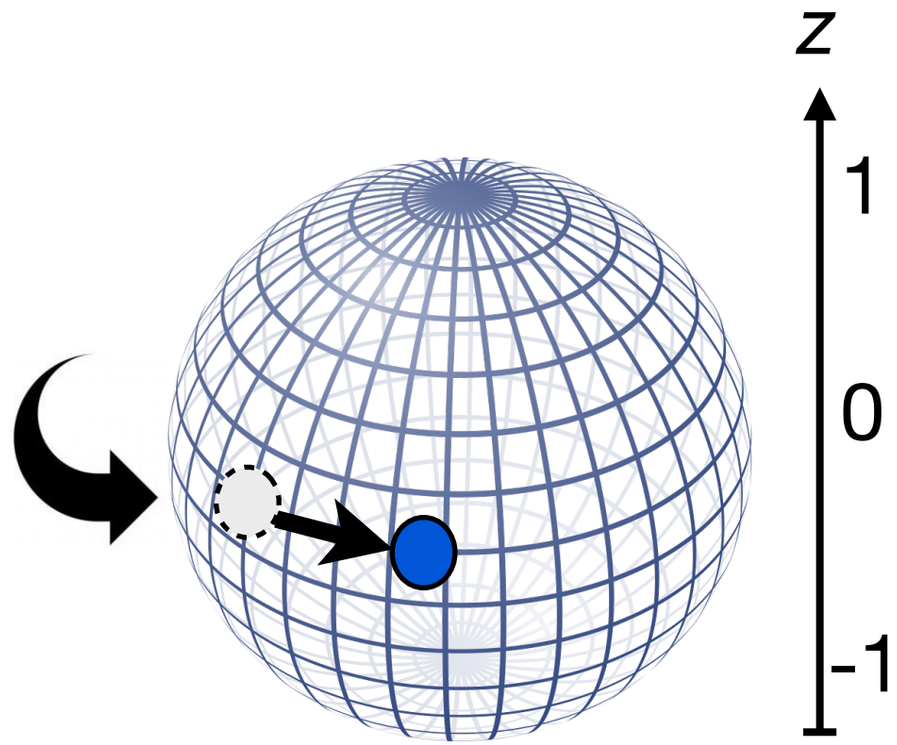
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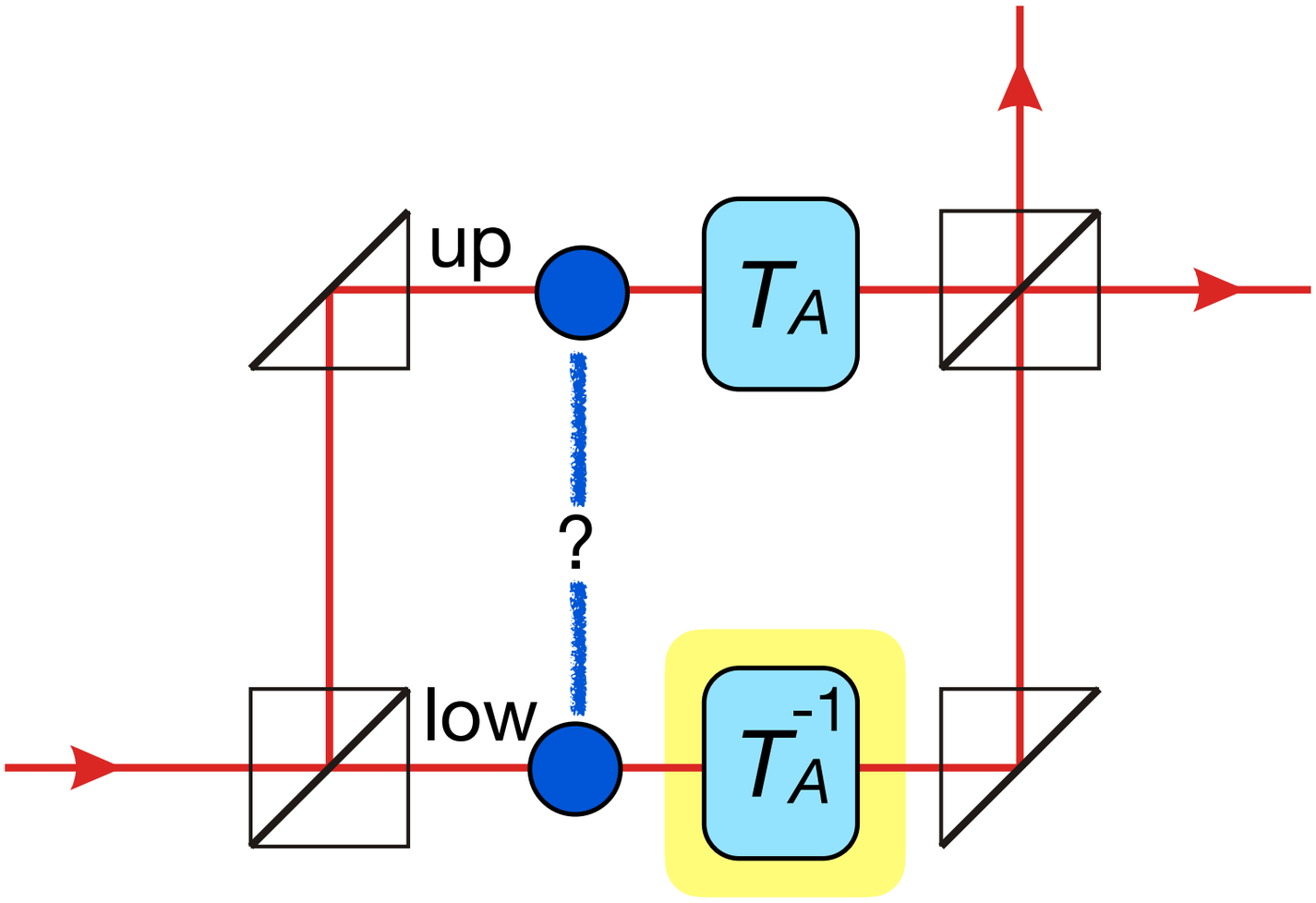
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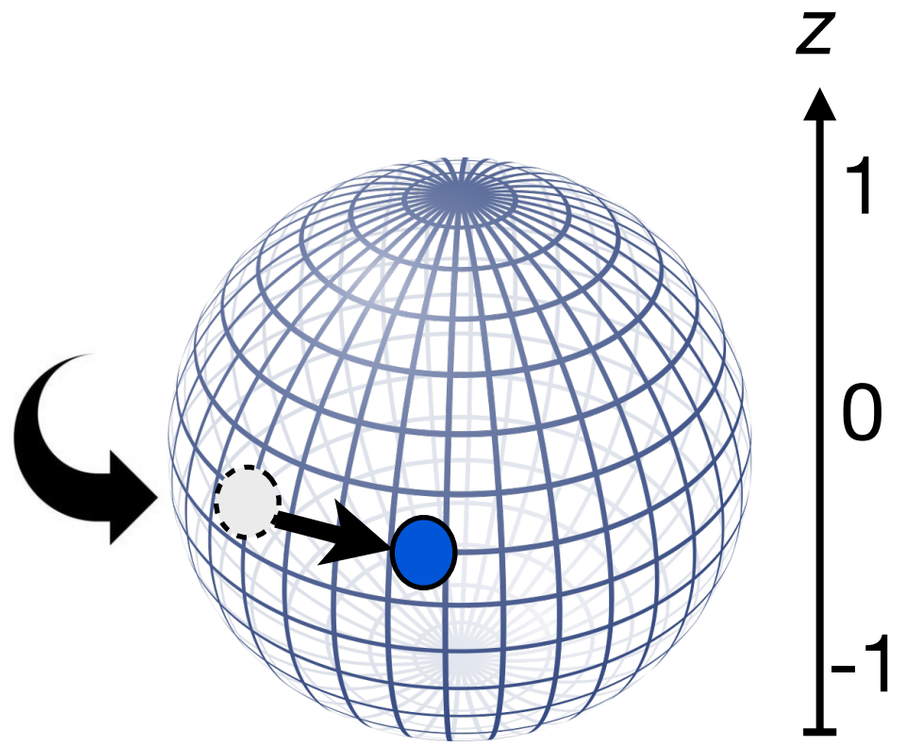




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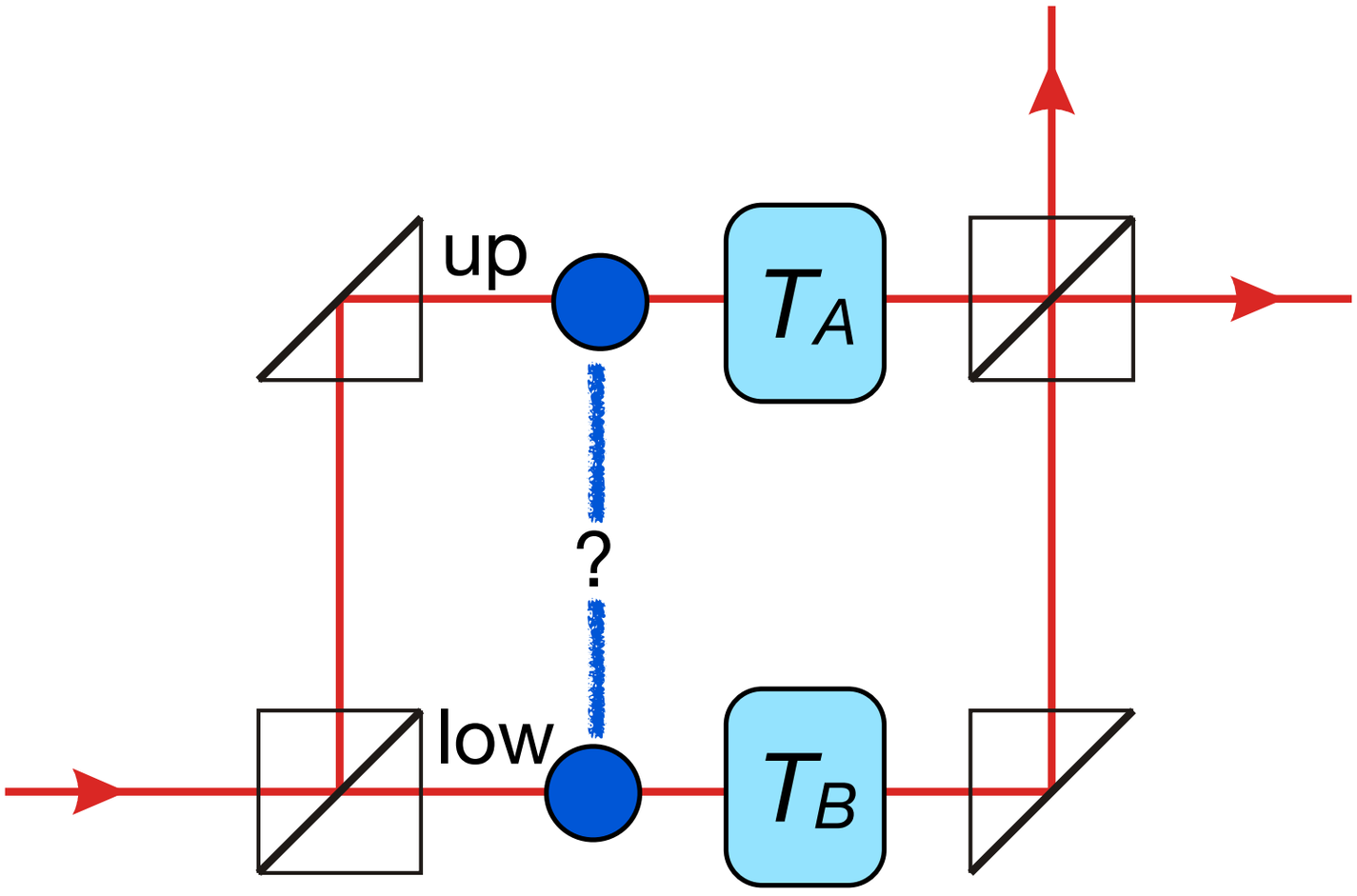


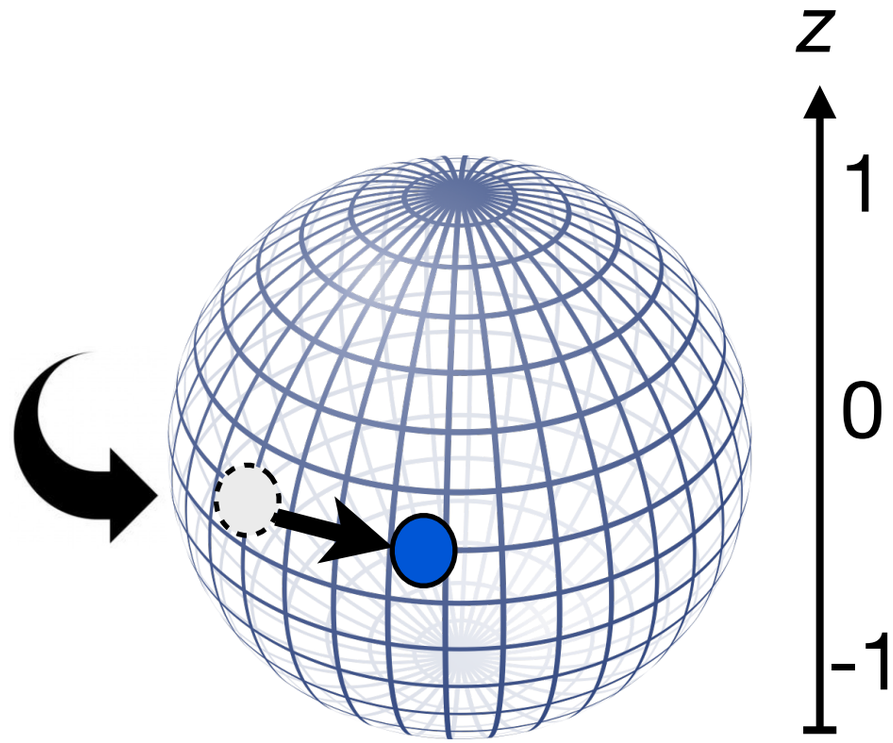
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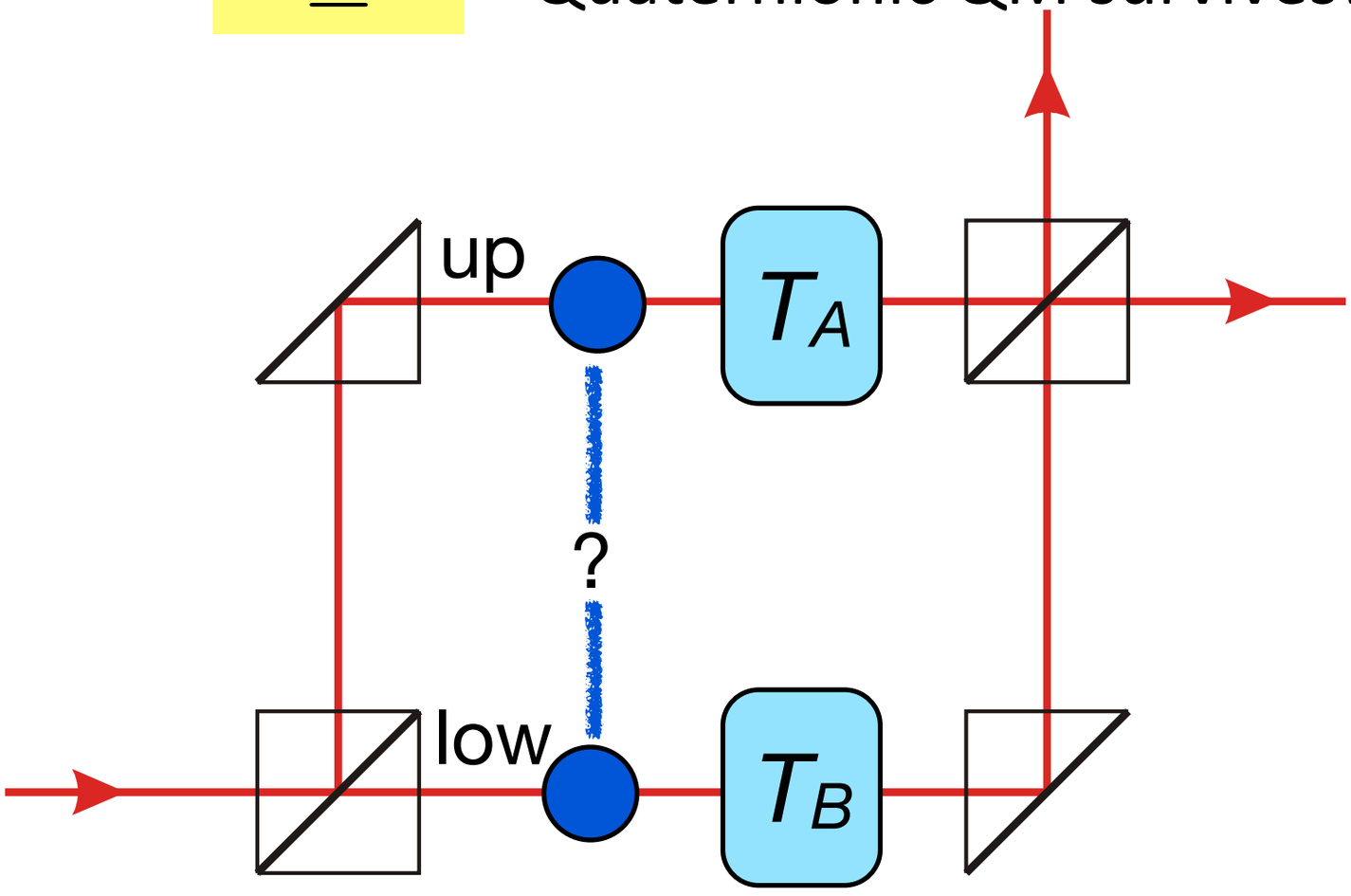
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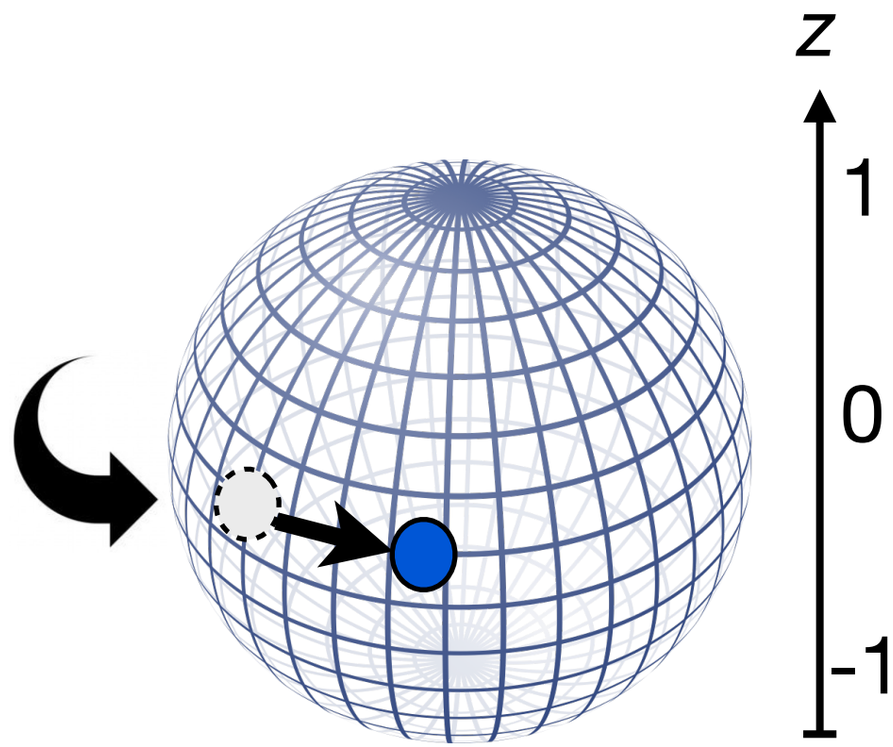
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$\Rightarrow d \leq 5$. Quaternionic QM survives!





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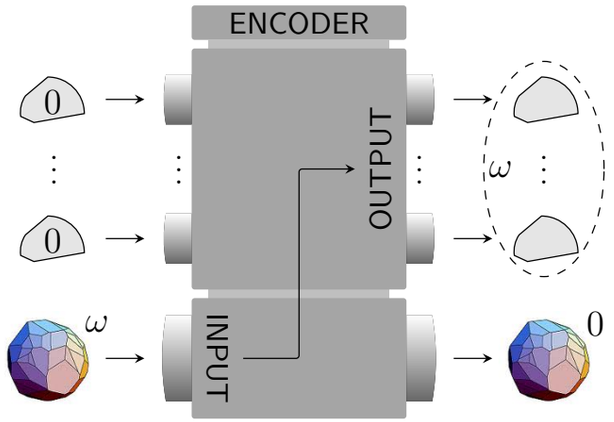
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Spacetime structure constrains our world's probabilistic theory!

Overview

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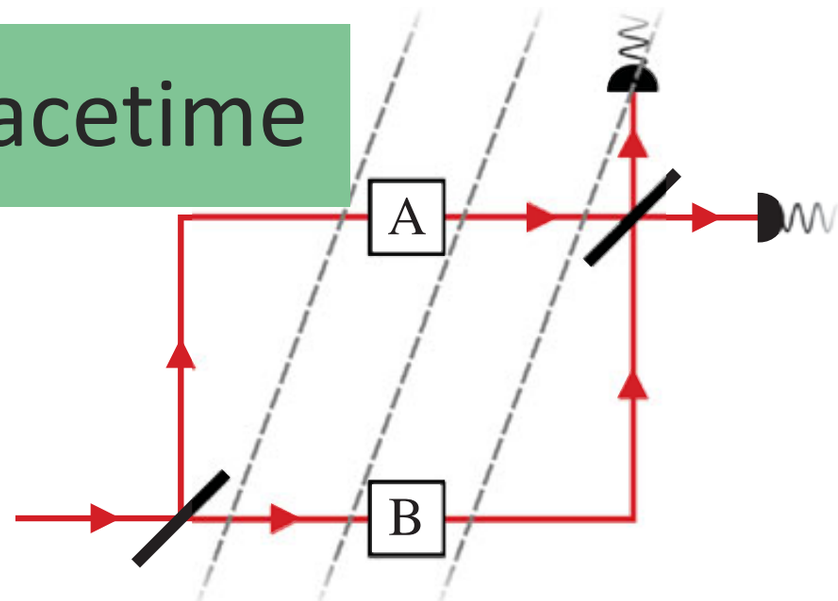
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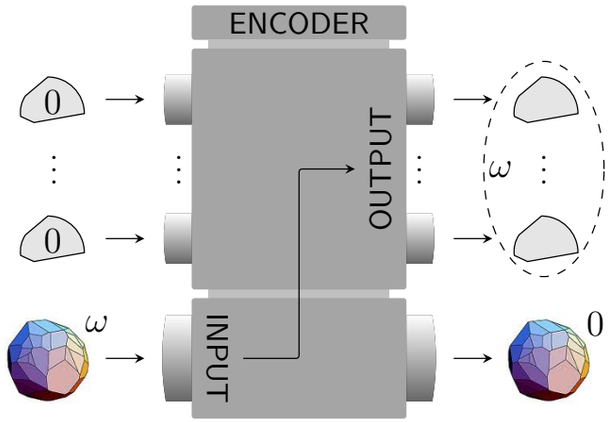
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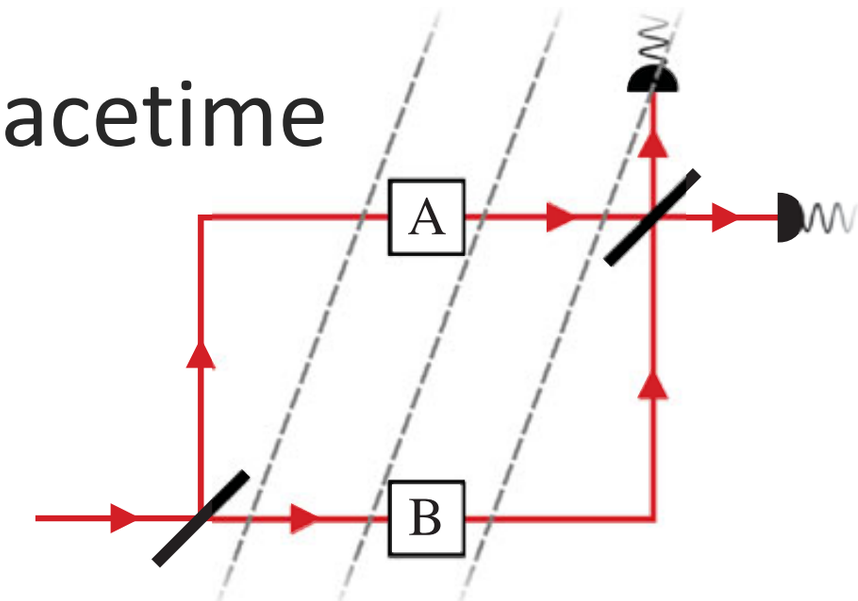
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PROCEEDINGS OF THE ROYAL SOCIETY A

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A glance beyond the quantum model

By Miguel Navascués[1,2,*] and Harald Wunderlich[1,2,3]

[1]*Institute for Mathematical Sciences, Imperial College London, London SW7 2PG, UK*
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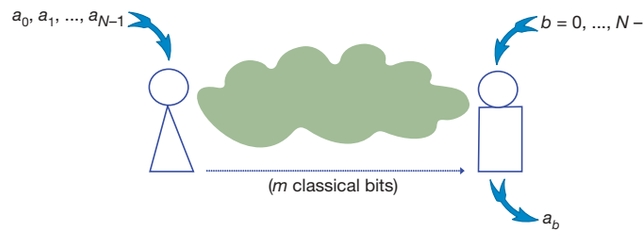
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